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МАТЕМАТИКА

П. И. ПЕТРОВ

КЛАССИФИКАЦИЯ КОНФОРМНО-ПЛОСКИХ РИЧЧИ-РЕКУРРЕНТНЫХ РИМАНОВЫХ МНОГООБРАЗИЙ

Представлено А. МОСТОВСКИМ 7 июля 1958 г.

1. Пусть C_n означает конформно-плоское риманово пространство V_n. Тензор кривизны R_{hijk} всякого C_n выражается формулой

$$R_{hijk} = g_{ik} L_{hj} - g_{ij} L_{hk} + g_{hj} L_{ik} - g_{hk} L_{ij}, \tag{1}$$

где g_{ik} — составляющие фундаментального тензора, L_{ij} — некоторого другого симметрического тензора валентности два. Между составляющими тензора Риччи $R_{ij} = g^{\lambda\mu} R_{i\lambda\mu j}$ и L_{ik}, $L = g^{\lambda\mu} L_{\lambda\mu}$ существует линейная зависимость

$$L_{ik} = \frac{1}{2-n}(R_{ik} + L g_{ik}). \tag{2}$$

Требование риччи-рекуррентности $V_n\,(n > 2)$ [1]

$$R_{\alpha\beta,\gamma} = k_\gamma R_{\alpha\beta}, \tag{3}$$

где $R_{\alpha\beta} \neq 0$, k_i — вектор, имеющий, по крайней мере, одну ненулевую составляющую, равносильно условию

$$L_{\alpha\beta,\gamma} = k_\gamma L_{\alpha\beta}. \tag{4}$$

Обозначим через R_n, как и в [1], n-мерное риччи-рекуррентное риманово многообразие.

Формула (1) не является характерной для C_3, так как тензор кривизны любого трехмерного пространства V_3 имеет указанный ею вид. Чтобы V_3 оказалась C_3, необходимо и достаточно, как известно

$$L_{\alpha[\beta,\gamma]} = 0. \tag{5}$$

При $n > 3$ дифференциальное уравнение (5) является следствием формулы (1) и тождества Бианки $R_{\alpha\beta[\gamma\delta,\varepsilon]} = 0$. Так как g_{ij} и R_{hijk} взаимно определяют друг друга [2], то задача классификации конформно-плоских риччи-рекуррентных пространств $C_n(R_n)$ приводится к вопросу нахождения в метрическом пространстве неэквивалентных типов

симметрических тензоров, составляющие которых L_{ij} одновременно удовлетворяют как алгебраическим

$$L_{ij}k_l - L_{il}k_j = 0, \tag{6}$$

так и дифференциальным соотношениям [4]. Чтобы решить последнюю задачу, ассоциируем с формой ds^2, устанавливающей мероопределение риманова пространства $C_n(R_n)$, арифметический инвариант — характеристику λ матрицы $\|L_{ij} - \lambda g_{ij}\|$ и условимся отнести к одному классу все ds^2 с совпадающими характеристиками, хотя не все формы таким образом выделенного класса дифференциальных форм будут эквивалентными. Непустота возникающих при этом классов пространств (форм ds^2), автоматически не вытекает из самого способа применяемого алгебраического метода, так как коэффициенты рассматриваемой здесь пары (L_{ij}, g_{ij}) связаны не только структурными соотношениями (4), (6), но и функциональной зависимостью — дифференциальной системой второго порядка, тогда как изучаемые в алгебре формы совершенно независимы между собою. Ниже дается эффективное решение этого вопроса.

2. Нашей ближайшей целью является доказательство нескольких вспомогательных предложений, которые помогут нам решить поставленную задачу.

Лемма 1. *В пространстве $C_n(R_n)$ симметрическая матрица*

$$\mathcal{L} = (L_{ik}) \qquad (i, k = 1, 2, \ldots, n) \tag{7}$$

имеет ранг $r = 1$.

Доказательство. Гипотеза $r = 2$ о натуральном числе $r > 1$ влечет за собой $k_i = 0$, что невозможно для $C_n(R_n)$. Значит, $r = 1$.

Лемма 2. *Вейерштрассова характеристика λ-матрицы $\mathcal{L} - \lambda g$ для неприводимых $C_n(R_n)$ выражается символом*

$$|(\underbrace{\mathring{1}\,\mathring{1}\ldots\mathring{1}}_{n-2}\,\mathring{2})|. \tag{8}$$

Доказательство. Все элементарные делители λ-матрицы $\|R_{\alpha\beta} - \lambda g_{\alpha\beta}\|$ в неприводимом R_n имеют вид λ, λ^2 (см. [1], стр. 291). Поэтому характеристика матрицы $\mathcal{L} - \lambda g$ состоит исключительно из единиц и двоек. По первой лемме двойка в нее может войти не более одного раза. Лемма 2 доказана.

3. Тернарная форма

$$ds^2 = e^{x_1+x_2+x_3}\,dx_1^2 + 2dx_1\,dx_2 + dx_3^2 \tag{9}$$

приводимая, риччи-рекуррентная, а полином $\varphi(\lambda) = |R_{\alpha\beta} - \lambda g_{\alpha\beta}|$, отвечающий ей, имеет два неравных нулю корня. Форма

$$ds^2 = 2\,e^{2x_1}(dx_1\,dx_2 + dx_3\,dx_4) + \sum_{i=5}^{n} \varepsilon_i dx_i^2;\ (\varepsilon_i = \pm 1), \tag{10}$$

хотя и приводимая, но все главные инварианты ее тензора Риччи равны нулю. Чтобы получить представление о структуре тех приводимых риччи-рекуррентных дифференциальных квадратичных форм, у которых полином $\varphi(\lambda)$ имеет отличный от нуля корень, обратим внимание на условия риччи-рекуррентности [3] формы

$$ds^2 = g_{\lambda\mu}(x^\nu)\,dx^\lambda\,dx^\mu + g_{\sigma\varrho}(x^\tau)\,dx^\sigma\,dx^\varrho, \tag{11}$$

где

$$\lambda, \mu, \nu = 1, 2, \ldots, r; \quad \sigma, \varrho, \tau = r+1, \ldots, n.$$

Так как символы Христоффеля, составляющие тензора кривизны и тензора Риччи равны нулю, когда у них имеются индексы принадлежащие к различным подпространствам, на которые разлагается данная форма, то эти условия для (11) будут иметь вид:

$$\begin{cases} 0 = R_{\lambda\mu,\,\tau} = R_{\lambda\mu}\cdot k_\tau, & R_{\lambda\mu,\,\nu} = k_\tau R_{\lambda\mu} \\ 0 = R_{\sigma\varrho,\,\nu} = R_{\sigma\varrho}\cdot k_\nu, & R_{\sigma\varrho,\,\tau} = k_\tau R_{\sigma\varrho}. \end{cases} \tag{12}$$

Отсюда, в силу требования $k_i \neq 0$, следует либо $R_{\lambda\mu} = 0$, либо $R_{\sigma\varrho} = 0$. Предположим, что $R_{\sigma\varrho} = 0$. Это означает, что матрица $\mathcal{R} = (R_{\alpha\beta})$ формы (11) изображается таблицей

$$\left(\begin{array}{ccc|c} R_{11} & R_{12} \ldots & R_{1r} & \\ \cdot & \cdot \cdot \cdot \cdot & \cdot & 0 \\ R_{r1} & \cdot \cdot & R_{rr} & \\ \hline & 0 & & 0 \end{array}\right). \tag{13}$$

Представим себе теперь этот процесс разбиения формы на независимые формы продолжающийся до тех пор, пока у нас не получится неприводимая риччи-рекуррентная форма от r_m независимых переменных. Таким образом, если метрическая форма ds^2 некоторого R_n приводимая, то существует такая система координат, относительно которой матрицы $g = (g_{ik})$, $\mathcal{R}(R_{ik})$ одновременно приводятся, соответственно, к видам

$$\left(\begin{array}{c|c} g_{r_m} & 0 \\ \hline 0 & g_{n-r_m} \end{array}\right), \quad \left(\begin{array}{c|c} \mathcal{R}_{r_m} & 0 \\ \hline 0 & 0 \end{array}\right). \tag{14}$$

Здесь g_{r_m} — матрица неприводимой риччи-рекуррентной дифференциальной квадратичной формы от r_m переменных, а $\mathcal{R}_{r_m}$ — матрица, составленная из компонент ее тензора Риччи. Вместе с тем это означает, что

$$\varphi(\lambda) = (-1)^{n-r_m}\, g_{n-r_m} \cdot \lambda^{n-r_m} \cdot \mathcal{R}_{r_m} - \lambda g_{r_m}. \tag{15}$$

Из формулы (15) можно извлечь полезные следствия.

Лемма 3. *Для того, чтобы полином $\varphi(\lambda)$ приводимого R_n имел корень $\lambda_1 \neq 0$, необходимо и достаточно, чтобы $r_m = 2$ и это V_2 имело непостоянную скалярную кривизну $R(x)$.*

Доказательство. Условие необходимо, так как в неприводимом $R_n (n > 2)$ все корни полинома $|R_{\alpha\beta} - \lambda g_{\alpha\beta}| = 0$ равны нулю (см. Теорему 2 статьи [1]). Условие достаточно, так как при $r_m = 2$

$$\varphi(\lambda) = (-1)^{n-2} |g_2| \cdot |g_{n-2}| \cdot \lambda^{n-2} \left(\frac{R}{2} - \lambda\right)^2, \tag{16}$$

где R — скалярная кривизна V_2, которая по условию отлична от нуля.

Е. М. Паттерсон доказал, что в неприводимом R_n все корни полинома $\varphi(\lambda) = 0$ равны нулю. Как показывает пример (10), теорема, обратная этой, вообще говоря, не верна. Однако, для $n = 3$ имеем.

Теорема 1. *R_3 тогда и только тогда неприводимое, если все корни характеристического полинома $\varphi(\lambda) = 0$ суть нули.*

4. Изложенные в пкт. 3 сведения о строении метрических форм приводимых R_n дают нам возможность доказать следующее предложение о пространствах $C_n(R_n)$:

Лемма 4. *Среди независимых форм, на которые разлагается метрическая форма приводимого $C_n(R_n)$, не может фигурировать бинарная форма непостоянной скалярной кривизны.*

Доказательство. Если бы утверждение Леммы 4 было неверно, то имели бы

$$\mathscr{L} = \begin{pmatrix} \dfrac{R}{2(1-n)} g_2 & 0 \\ 0 & \dfrac{R}{2(1-n)(2-n)} g_{n-2} \end{pmatrix}. \tag{17}$$

откуда получили бы, что $|\mathscr{L}| \neq 0$. Но этот вывод противоречит предпосылке, что рассматриваемое пространство есть $C_n(R_n)$.

Следствие 1. *$C_3(R_3)$ — неприводимое.*

Следствие 2. *λ-матрица $\|R_{\alpha\beta} - \lambda g_{\alpha\beta}\|$ приводимого $C_n(R_n)$ имеет элементарные делители:*

$$\underbrace{\lambda, \lambda, \ldots, \lambda}_{n-2}, \lambda^2$$

5. Нам надлежит теперь решить вопрос о существовании $C_n(R_n)$. Для этого, очевидно, достаточно указать частные примеры дифференциальных квадратичных форм, дающих мероопределение этих пространств. Такими являются, например, формы

$$ds^2 = x_1 x_3^2 dx_1^2 + 2 dx_1 dx_2 + dx_3^2, \tag{18}$$

$$ds^2 = 2 e^{2x_1} (dx_1 dx_2 + dx_3 dx_4), \tag{19}$$

$$ds^2 = e^{2x_5}\left(2\,dx_1\,dx_2 + 2\,dx_3\,dx_4 + \sum_{i=5}^{n} \varepsilon_i\,dx_i^2\right) \quad (\varepsilon_i = \pm 1). \tag{20}$$

Итак, доказана

ЛЕММА 5. *Для любого натурального* $n > 2$ *существует* $C_n(R_n)$.

Учитывая вышеприведенные факты, имеем право высказать теорему:

ТЕОРЕМА 2. *Риманово многообразие* $C_n(R_n)$ *характеризуется арифметическим инвариантом* $\underbrace{[(\mathring{1}\,\mathring{1}\,\ldots\,\mathring{1}\,\mathring{2})]}_{n-2}$ *своего тензора* L_{ik}.

6. Пусть K_n символизирует рекуррентное риманово пространство [3].

ЛЕММА 6. $C_n(R_n) \subset K_n$.

Доказательство. Формула (1) и определение пространства K_n.

Рюз обнаружил [4] существование рекуррентного просто гармонического (simply harmonic) риманового многообразия H_n. Спрашивается, имеются-ли среды $C_n(R_n)$ пространства H_n? Ответ на поставленный вопрос дается теоремой.

ТЕОРЕМА 3. $C_n(R_n) \cap H_n = \varnothing$.

Докажем сначала одну вспомогательную теорему:

ЛЕММА 7. *Если след матрицы*

$$\Gamma = (\Gamma_j^i) \qquad i, j = 1, 2, \ldots, n,$$

образованной из величин

$$\Gamma_j^i = R_{pqj}^i\,\xi^p\,\xi^q, \tag{21}$$

где ξ^i *— произвольный вектор,* R_{pqj}^i *— составляющие тензора кривизны, для пространства* $C_n(R_n)$ *равен нулю, то оно плоское.*

Доказательство. На основании формул (1), (21) имеем

$$\Gamma_j^i = \xi_j\,\xi^q\,L_q^i - g_{pq}\,\xi^p\,\xi^q\,L_j^i + \xi^p\,\xi^i\,L_{pj} - \delta_j^i\,L_{pq}\,\xi^q\,\xi^p. \tag{22}$$

Контрактируя (22) и учитывая, что по Теореме 2 в $C_n(R_n)$ $L^\sigma\,\sigma = 0$, находим

$$\Gamma^\sigma\,\sigma = (2 - n)\,L_{ij}\,\xi^i\,\xi^j. \tag{23}$$

Следовательно, из условия $\Gamma^\sigma\,\sigma = 0$ при $n \neq 2$ следует $L_{ij} = 0$; значит, и $R_{hijk} = 0$.

Приступим теперь к доказательству Теоремы 3. Если бы $C_n(R_n)$ оказалось просто гармоническим, то — согласно критерию Рюза — все корни характеристического многочлена матрицы Γ были бы равны нулю. Это означало бы в то же время и исчезновение ее следа. Тогда по Лемме 7 пространство $C_n(R_n)$ было бы плоским, а плоские простран-

ства не входят в число риччи-рекуррентных. Полученное противоречие доказывает справедливость заключения Теоремы 3 о том, что $C_n(R_n)$ не может быть просто гармоническим.

7. Отыскание скалярных дифференциальных инвариантов второго порядка риманового пространства на основании теоремы приведения [5] для тензора g_{ik} сводится к разысканию совокупных алгебраических инвариантов двух тензоров g_{ik}, R_{hijk}. Для $C_n(R_n)$ эта задача, в свою очередь, сводится к определению инвариантов пары форм (L_{ik}, g_{ik}). По Теореме 2 эта пара не имеет алгебраического инварианта. Отсюда получается

ТЕОРЕМА 4. *Риманово пространство $C_n(R_n)$ не имеет скалярного дифференциальнго инварианта второго порядка.*

ИНСТИТУТ МАТЕМАТИКИ КАЗАНСКОГО УНИВЕРСИТЕТА (СССР)

ЦИТИРОВАННАЯ ЛИТЕРАТУРА

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P. I. PETROV, CLASSIFICATION DES VARIÉTÉS RIEMANNIENNES CONFORMÉMENT PLATES RICCI-RÉCURRENTES

Soit ds^2 la forme métrique d'espace $C_n(R_n)$ indiquée dans le titre de cette note. Associons l'invariant arithmétique qui est la caractéristique de λ-matrice $\|L_{ik} - \lambda g_{ik}\|$ avec ds^2 où L_{ik} satisfait aux relations (1), (4), (6).

Dans notre travail cet invariant arithmétique est employé dans le but de classification des espaces $C_n(R_n)$:

1°. La caractéristique de $C_n(R_n)$ est

$$|(\underbrace{\mathring{1}\,\mathring{1}\ldots\mathring{1}}_{n-2}\,\mathring{2})|.$$

En plus de la conclusion essentielle 1° dans cette note les théoremes suivants sont démontrés:

2°. *L'espace ricci-recurrent R_3 est irréducible si et seulement si l'équation $|R_{ij} - \lambda g_{ij}| = 0$, (où R_{ij} est le tenseur de Ricci) possède toutes les racines égales à zéro.*

3°. *$C_n(R_n)$ n'est jamais un espace simplement harmonique.*

4°. *Les éspaces riemanniens $C_n(R_n)$ ne possèdent pas un invariant différentiel scalaire de deuxième ordre.*

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MATHÉMATIQUE

Sur les involutions d'ordre n

par

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Présenté par A. MOSTOWSKI le 10 octobre 1960

Soit X un espace linéaire sur le corps des nombres complexes. Nous appellerons une opération algébrique linéaire — *involution d'ordre n*, si n est le plus petit nombre naturel, satisfaisant à l'égalité suivante

$$S^n = I. \tag{1}$$

I désignant l'identité.

Dans le travail [2] nous avons présenté quelques exemples des involutions d'ordre 2. La transformation de Fourier des fonctions définies sur un espace euclidien E^k de dimension k, intégrables et intégrables avec leurs carrés, est une *involution d'ordre* 4. Evidemment, en désignant

$$\varphi(x) = Tf(y) = \frac{1}{(2\pi)^{k/2}} \int_{E^k} f(y)\, e^{i(x,y)}\, dy,$$

$$f(y) = \overline{T}\,\varphi(x) = \frac{1}{(2\pi)^{k/2}} \int_{E^k} \varphi(x)\, e^{-i(y,x)}\, dx$$

(d'après le travail [1]), nous avons: $T\overline{T} = I$ et

$$f(-y) = T\overline{T}f(-y) = T(-1)^{2k}\varphi(x) = T^2 f(y), \tag{2}$$

d'où

$$T^4 f(y) = T^2(T^2 f(y)) = T^2 f(-y) = f(y)$$

et finalement

$$T^4 = I. \tag{3}$$

Si les fonctions $f(y)$ sont des fonctions paires, c-à-d. si $f(-y) = f(y)$, alors la transformation de Fourier, d'après l'égalité (2), est une involution d'ordre 2:

$$T^2 = I. \tag{4}$$

Dans ce travail nous allons présenter une généralisation de quelques théorèmes de notre travail précedent [2].

Soit S une involution d'ordre n. Désignons

$$\varepsilon = e^{2\pi i/n}; \quad \varepsilon_\nu = \varepsilon^\nu \quad (\nu = 1, 2, \ldots, n). \tag{5}$$

Nous avons évidemment:

$$\varepsilon_\nu^n = \varepsilon_n^\nu = 1, \tag{6}$$

$$\varepsilon_\nu^{-k} = \varepsilon_\nu^{-k} = \varepsilon_\nu^{n-k} = \varepsilon_{n-k}^\nu \qquad (\nu = 1, 2, \ldots, n), \tag{6'}$$

$$\frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_\nu^k = \begin{cases} 1, & \text{pour } k = 0 \\ 0, & \text{pour } k \pm 1, \pm 2, \ldots \end{cases} \tag{6''}$$

Désignons encore

$$P_\nu = I + \varepsilon^{-\nu} S + \ldots + \varepsilon^{-\nu(n-1)} S^{n-1}. \tag{7}$$

On aura évidemment

$$(I - \varepsilon^{-\nu} S) P_\nu = I - S^n = 0, \tag{8}$$

$$\frac{1}{n}\sum_{\nu=1}^{n} P_\nu = I. \tag{8'}$$

Soient les ensembles suivants:

$$x^{(\nu)} = P_\nu X \qquad (\nu = 1, 2, \ldots n). \tag{9}$$

Donc, en vertu de (8'), tout élément $x \in X$ peut être présenté de la manière suivante:

$$x = \sum_{\nu=1}^{n} x^{(\nu)}, \tag{10}$$

où

$$x^{(\nu)} = \frac{1}{n} P_\nu x \in X. \tag{11}$$

Remarquons, que — d'après l'égalité (8) —

$$Sx^{(\nu)} = \varepsilon^\nu x^{(\nu)}, \qquad \text{pour } x^{(\nu)} \in X^{(\nu)}. \tag{12}$$

Il en résulte l'indépendence linéaire des sous-espaces $X^{(\nu)}$.

Au contraire, si tout élément $x \in X$ peut être présenté sous la forme d'une somme des élements linéaires indépendants $x^{(\nu)}$; $x = \sum_{\nu=1}^{n} x^{(\nu)}$, tels, que $Sx^{(\nu)} = \varepsilon^{-\nu} x^{(\nu)}$, alors, d'après l'égalité (6''), nous avons

$$\frac{1}{n} P_\nu x = \frac{1}{n}\left(I + \varepsilon^{(\nu)} S + \ldots + \varepsilon^{-\nu(n-1)} S^{n-1}\right), \sum_{k=1}^{n} x^{(k)} =$$

$$= \frac{1}{n}\sum_{k=1}^{n} (1 + \varepsilon^{k-\nu} + \ldots + \varepsilon^{(k-\nu)(n-1)}) x^{(k)} =$$

$$= \frac{1}{n}\sum_{k=1}^{n} \left(\varepsilon_0^{k-\nu} + \varepsilon_1^{k-\nu} + \ldots + \varepsilon_{n-1}^{k-\nu}\right) x^{(k)} = x^{(\nu)}.$$

Il en résulte le théorème suivant

THÉORÈME 1. *L'espace X est une somme directe des espaces $X^{(\nu)}$ ($\nu = 1, 2, \ldots, n$) linéaires indépendants, tels, que*

$$Sx = \varepsilon^{-\nu} x \quad \text{pour} \quad x \in X^{(\nu)}.$$

Considérons une équation *involutive* avec les coefficients constants (complexes):

$$(a_0 + a_1 S + \ldots + a_{n-1} S^{n-1})\, x = x_0, \tag{13}$$

où $x_0 \in X$ est donné et $\sum_{\nu=0}^{n-1} |a_\nu| > 0$.

Désignons

$$b_\nu = a_0 + a_1 \varepsilon_\nu + \ldots + a_{n-1} \varepsilon_\nu^{n-1} \quad (\nu = 1, 2, \ldots, n). \tag{14}$$

Alors, d'après le Théorème 1 et l'égalité (12), l'équation (13) est équivalente au système des équations suivantes:

$$b_\nu x^{(\nu)} = x_0^{(\nu)} \quad (\nu = 1, 2, \ldots, n) \tag{15}$$

où

$$x_0^{(\nu)} = \frac{1}{n} P_\nu x_0 \in X^{(\nu)}, \qquad x^{(\nu)} \in X^{(\nu)}.$$

Le degré de polynôme de la variable complexe

$$b(t) = a_0 + a_1 t + \ldots + a_{n-1} t^{n-1} \tag{16}$$

est au plus égal à $n - 1$. Donc, l'équation $b(t) = 0$ admet au plus $(n - 1)$ de racines complexes. Mais il y a n nombres ε_ν (différents), d'où il résulte, qu'il existe au moins un indice ν_0 tel, que

$$b_{\nu_0} = b_{(\nu_0)} \neq 0.$$

THÉORÈME 2. *Si $b_\nu \neq 0$ pour $\nu = 1, 2, \ldots, n$, alors l'équation* (13) *homogène ($x_0 = 0$) n'admet qu'une solution nulle, l'équation* (13) *pour $x_0 \neq 0$ arbitraire admet une solution unique*

$$x = \left(\sum_{\nu=1}^{n} b_\nu^{-1} P_\nu \right) x_0. \tag{17}$$

Démonstration. Si $b_\nu \neq 0$ et $x_0 = 0$, alors, d'après l'équation (15) il suffit de poser $x^{(\nu)} = 0$ pour $\nu = 1, 2, \ldots, n$, d'où $x = 0$. Si $b_\nu \neq 0$ et $x_0 \neq 0$, alors le système (15) admet une solution unique et

$$x = \sum_{\nu=1}^{n} x^{(\nu)} = \sum_{\nu=1}^{n} b_\nu^{-1} x_0^{(\nu)} = \left(\sum_{\nu=1}^{n} b_\nu^{-1} P_\nu \right) x_0,$$

d'où résulte la thèse du Théorème 2.

THÉORÈME 3. *Si pour $\nu = k_i$ ($i = 1, 2, \ldots, m = n - 1$)*

$$b_{k_i} = 0 \tag{18}$$

alors l'équation (13) *homogène admet la solution*

$$y = \sum_{i=1}^{m} y^{(k_i)} \qquad (y^{(k_i)} \in X^{(k_i)} \text{ est arbitraire}) \tag{19}$$

l'équation (13) $(x_0 \neq 0)$ *admet a solution*

$$x = \left(\sum_{\substack{\nu=1 \\ \nu \neq k_i \ (i=1,2,\ldots,m)}}^{n} b_\nu^{-1} P_\nu \right) x_0 + y \tag{20}$$

(*où* y *est la solution* (19) *de l'équation homogène*) *sous la condition nécessaire et suffisante*

$$P_{k_i} x_0 = 0 \quad (i = 1, 2, \ldots, m). \tag{21}$$

Démonstration. Si $b_{k_i} = 0$ et $x_0 = 0$, alors pour $\nu \neq k_i$ il suffit de poser $x^{(\nu)} = 0$; pour $\nu = k_i$: $x^{(k_i)} = y^{(k_i)}$ arbitraire. Si $b_{k_i} = 0$ et $x_0 \neq 0$, alors il suffit de poser $x_0^{(k_i)} = P_{k_i} x_0 = 0$ et $x^{(k_i)}$ peut être arbitraire. Pour $\nu \neq k_i$ nous avons évidemment

$$x^{(\nu)} = b_\nu^{-1} x_0^{(\nu)} = b_\nu^{-1} P_\nu x_0,$$

d'où la solution (20).

Si, au contraire x est une solution de l'équation (13), nous avons $b_{k_i} = 0$ pour $i = 1, 2, \ldots, m$, d'où

$$P_{k_i} x_0 = x^{(k_i)} = b_{k_i} x^{(k_i)} = 0 \tag{22}$$

donc la condition (21) est non seulement suffisante mais aussi nécessaire.

Supposons maintenant que l'espace X est unitaire (c.-à.-d. un espace linéaire sur le corps des nombres complexes avec un produit scalaire).

Théorème 4. *Si l'opération* S *algébrique linéaire* (*qui transforme l'espace* X *en lui-même*) *est unitaire, si, en outre,* k *et* m *sont les plus petits nombres naturels tels que*

$$(S^*)^k = S^m \tag{23}$$

(*ou* S^* *désigne une opération adjointe avec l'opération* S), *alors* S *est une involution d'ordre* $m+n$.

Démonstration. D'après les hypothèses pour $x, y \in X$ arbitraires nous avons:

$$(S^{k+m} x, y) = (S^m x, (S^k)^* y) = (S^m x, (S^*)^k y) = (S^m x, S^m y) = (x, y),$$

d'où

$$S^{k+m} = I.$$

Théorème 5. *Pour que les sous-espaces* $X^{(k)}$ *et* $X^{(k')}$ $(k, k' = 1, 2, \ldots, n; k \neq k')$ *soient orthogonaux, il faut, que l'involution* S *soit une isométrie*

$$\|Sx\| = \|x\| \quad (\text{où } \|x\| = (x, x)^{1/2}). \tag{24}$$

Démonstration. Soit $X^{(k)}$ orthogonal à $X^{(k')}$, donc

$$(x^{(k)}, x^{(k)}) = (x^{(k')}, x^{(k)}) = 0 \quad (k, k' = 1, 2, \ldots, n; \ k \neq k').$$

Alors

$$\|Sx\|^2 = (Sx, Sx) = \left(\sum_{\nu=1}^{n} x^{(\nu)}, \sum_{\nu=1}^{n} x^{(\nu)} \right) = \sum_{\nu=1}^{n} (\varepsilon_\nu x^{(\nu)}, \varepsilon_\nu x^{(\nu)}) =$$

$$= \sum_{\nu=1}^{n} \varepsilon_\nu \bar{\varepsilon}_\nu (x^{(\nu)}, x^{(\nu)}) = \sum_{\nu=1}^{n} (x^{(\nu)}, x^{(\nu)}) = (x, x) = \|x\|^2$$

d'après la formula (6'). Il en résulte la thèse du Théorème 5.

Si nous définissons un nouveau produit scalaire par la formule suivante:

$$(x,y)_S = \sum_{\nu=1}^{n} (S^{(\nu)} x, S^{(\nu)} y), \tag{25}$$

alors on voit, que l'involution S est une opération unitaire au sens du produit (25), donc elle est aussi une isométrie. L'opération adjointe (au sens du produit (25)) à l'opération S^k ($k = 1, 2, \ldots, n$) est l'opération suivante:

$$(S^k)^*_S = S^{n-k}. \tag{26}$$

En réalité, nous avons

$$(S^k x, y)_S = (S^k x, S^n y)_S = (x, S^{n-k} y)_S.$$

THÉORÈME 6. *Pour $k, k' = 1, 2, \ldots, n$, $k \neq k'$; les sous-espaces $X^{(k)}$ et $X^{(k')}$ sont orthogonaux au sens du produit* (25).

Démonstration. Soient $x \in X^{(k)}$, $y \in X^{(k')}$ arbitraires. Alors nous avons (d'après (6'), (6'') et (12)):

$$(x, y)_S = \sum_{\nu=1}^{n} (S^\nu x, S^\nu y) = \sum_{\nu=1}^{n} (\varepsilon_k^\nu x, \varepsilon_{k'}^\nu y) = \sum_{\nu=1}^{n} \varepsilon_k^\nu \varepsilon_{k'}^\nu (x, y) =$$

$$= (x, y) \sum_{\nu=1}^{n} (\varepsilon_k \varepsilon_{k'})^\nu = (x, y) \sum_{\nu=1}^{n} (\varepsilon^{k-k'})^\nu = (x, y) \sum_{\nu=1}^{n} \varepsilon_\nu^{k-k'} = 0,$$

d'où résulte la thèse du Théorème 6.

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MATHÉMATIQUE

Sur les équations involutives d'ordre n

par

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Présenté par A. MOSTOWSKI, le 18 octobre 1960

Considérons une équation involutive d'ordre n:

$$(A_0+A_1 S+\ldots+A_{n-1} S^{n-1}+T_A)\, x = x_0, \tag{1}$$

où x, x_0 sont des éléments d'un espace linéaire (sur le corps des nombres complexes) X, A_i $(i = 0, 1, \ldots, n-1)$ sont des opérations algébriques linéaires qui transforment l'espace X en lui-même, S est une *involution d'ordre n*, c.-à-d. une opération algébrique linéaire telle que $S^n = I$ (où I désigne l'identité, [3]), T_A désigne une opération linéaire pour le moment non déterminée.

Soient les opérations A_i commutatives entre elles. Désignons par A et K les opérations suivantes:

$$A = \sum_{i=0}^{n-1} A_i S^i; \qquad K_A = A+T_A \tag{2}$$

par D_A — le *déterminant* cyclique opérationnel suivant:

$$D_A = \begin{vmatrix} A_0, & A_1, \ldots, A_{n-1} \\ A_{n-1}, & A_0, \ldots, A_{n-2} \\ \cdots & \cdots \\ A_1, & A_2, \ldots, A_0 \end{vmatrix} \tag{3}$$

Soit donné un idéal J des opérations, c.à.d. un ensemble des opérations, qui satisfait aux conditions suivantes: si les opérations $T_1, T_2 \in J$ sont arbitraires et si A est une opération linéaire quelconque, si, enfin, a, b sont des nombres complexes, alors

$$aT_1+bT_2 \in J, \qquad T_1 A \in J, \qquad AT_1 \in J.$$

Soit $T_A \in J$. On appelle *régularisateur gauche* (*droit*) de l'opération K_A relativement à l'ideal J une opération $K_B = B+T_B$ (où $B = \sum_{i=0}^{n-1} B_i S^i$) telle que les opérations B_i algébriques linéaires sont commutatives entre elles, $(T_B \in J)$ à condition que

$$K_A K_B = I+T_g, \quad \text{où} \quad T_g \in J \quad (\text{ou } K_A K_B = I+T_d, \quad \text{où} \quad T_d \in J). \tag{4}$$

Si le régularisateur gauche est égal au régularisateur droit, alors on l'appelle — *régularisateur simple*. Donc, si R est un régularisateur simple, alors, en désignant

$$(5)\quad \begin{cases} K' = R_A K_A, \qquad K'_C = C' + T_{C'}, \qquad C' = \sum_{i=0}^{n-1} C'_i S^i, \\ K'' = K_A R_A, \qquad K''_C = C'' - T_{C''}, \qquad C'' = \sum_{i=0}^{n-1} C''_i S^i. \end{cases}$$

nous avons

$$(5')\quad C'_0 = C''_0 = I, \qquad C'_i = C''_i = 0 \ (i = 1, 2, \ldots, n-1),$$

$$T_{C'} \in J, \qquad T_{C''} \in J.$$

THÉORÈME 1. *Si l'opération K_A admet les propriétés suivantes*

(1) les opérations A_i sont commutatives, entre elles,

(2) le déterminant D_A admet un déterminant inverse D_A^{-1},

(3) $T_A \in J$ (où J désigne un idéal des opérations),

(4) $SA_i - A_i S \in J$ $(i = 0, 1, \ldots, n-1)$,

alors il existe un régularisateur simple R_A de l'opération K relativement à l'idéal J. Nous avons, en outre,

$$(6)\quad R_A = D_A^{-1} A'$$

où

$$(6')\quad A' = \sum_{i=0}^{n-1} A'_i S^i; \qquad A'_i = (-1)^i D_{A,i+1} \ (i = 0, 1, \ldots, n-1)$$

et $D_{A,i+1}$ désigne le sous-déterminant obtenu du déterminant D par effacement de la première colonne et la $(i+1)$-ième ligne de ce déterminant; en outre nous avons:

$$(7)\quad R_A K_A = I + T'_R; \qquad K_A R_A = I + T''_R$$

où

$$(7')\quad \begin{aligned} T'_R &= D_A^{-1}\left[\sum_{i,j=0}^{n-1} (-1)^j D_{A,i+1}(S^i A_j - A_j S^i) S^j + A T_A\right] \in J; \\ T''_R &= D_A^{-1}\left[\sum_{i,j=0}^{n-1} (-1)^j A_i (S^i D_{A,j+1} - D_{A,j+1} S^i) S^j + T_A A'\right] \in J. \end{aligned}$$

Démonstration. Désignons par $T_{k,i} = S^k A_i - A_i S^k$ $(k = 1, 2, \ldots, i = 0, 1, \ldots, n-1)$. Alors en vertu des hypothèses (4) $T_{1,i} \in J$ et nous avons evidemment

$$(8)\quad T_{k,i} = S^k A_i - A_i S^k = S^k A_i - S A_i S^{k-1} + S A_i S^{1-1} - A_i S^k = \\ = S T_{k-1,i} + T_{1,i} S^{k-1},$$

d'où, par l'induction mathématique, il résulte que

$$(8')\quad T_{k,i} \in J \qquad (k = 1, 2, \ldots).$$

Donc, on peut présenter chaque produit $B_i S^i A_j S^j$ (pour B_i arbitraire de la manière suivante

$$(9) \qquad B_i S^i A_j S^j = B_i A_j S^{i+j} + B_i (S^i A_j - A_j S^i) S^j = B_i A_j S^{i+j} + B_i T_{i,j} S^j.$$

Considérons une superposition des opérations $K_B = B+T$ et $K = A+T_A$. En désignant

$$K_C = K_B K_A; \qquad K_C = C+T_C; \qquad C = \sum_{i=0}^{n-1} C_i S^i$$

nous avons

$$(10) \qquad K_C = C+T_C = (B+T_B)(A+T_A) = BA+BT_A+T_B A+T_B T_A =$$

$$= \sum_{i,j=0}^{n-1} B_i S^i A_j S^j + BT_A + T_B A + T_B T_A = \sum_{i,j=0}^{n-1} B_i A_j S^{i+j} +$$

$$+ \sum_{i,j=0}^{n-1} B_i T_{i,j} S^j + BT_A + T_B A + T_B T_A,$$

d'où, d'après (8′) et (9),

$$(11) \qquad T_C = \sum_{i,j=0}^{n-1} B_i (S^i A_j - A_j S^i) S^j + BT_A + T_B A + T_B T_A \in J$$

et (en admettant $j = k - i$, $A_{n+k} = A_k$)

$$(12) \qquad C_k = \sum_{i=0}^{n-1} B_i A_{n+k-i} \qquad (k = 0, 1, \ldots, n-1).$$

Mais, d'après les propriétés des déterminants, nous avons:

$$(13) \qquad \sum_{i=0}^{n} (-1)^{i+k} D_{A,i} A_{n+k-i} = \begin{cases} D_A, & \text{si } k = 1 \\ 0, & \text{si } k = 2, 3, \ldots, n \end{cases}$$

Donc, si nous admettons

$$B = R_A = D_A^{-1} A'; \qquad T_B = 0,$$

où l'opération A' est déterminée par les formules (6′), (6″), nous aurons, d'après les formules (11), (12), (13)

$$R_A K_A = D_A^{-1} A' (A+T_A) = D_A^{-1} A' A + D_A^{-1} AT =$$

$$= D_A^{-1} \left(D_A + \sum_{i,j=0}^{n-1} (-1)^i D_{A,i+1} (S^i A_j - A_j S^i) S^j + A' T_A \right) = I + T_R',$$

où $T_R' \in J$.

Par analogie nous obtenons, d'après la propriété (13), les formules pour l'opération $K_A R_A$. Il en résulte la thèse du Théorème 1.

Soit X un espace de Banach. Soit J un certain idéal des opérations de Fredholm [2], par exemple un idéal des operations complètes-continues.

THÉORÈME 2. *Si K_A vérifie les suppositions du Théorème* 1 *et J est un certain idéal des opérations de Fredholm, alors*

1) le nombre k de solutions linéaires indépendantes de l'équation

$$K_A x = 0 \tag{14}$$

est fini;

2) la condition nécessaire et suffisante, pour que l'équation

$$K_A x = x_0 \tag{15}$$

ait une solution, est que

$$y_i(x_0) = 0 \quad (i = 1, 2, ..., k^*), \tag{16}$$

où y_i $(i = 1, 2, ..., k^*)$ désignent toutes les fonctionnelles linéaires indépendantes, qui sont des solutions de l'équation adjointe homogène

$$K_A^* y = 0; \tag{17}$$

3) *la différence du nombre k des solutions linéaires indépendantes de l'équation* (15) *et du nombre k^* des solutions linéaires indépendantes de l'équation* (17):

$$\varkappa = k - k^*$$

ne dépend que de A_i $(i = 0, 1, ..., n-1)$ et de S.

Démonstration. Elle résulte du Théorème 1 de la même manière que les démonstrations des Théorèmes (12), (13) et (14) du travail [2].

Dans le cas particulier nous obtenons de la démonstration du Théorème 1 un théorème algébrique pour l'équation involutive dans un espace linéaire X (sans l'hypothèse que l'espace X est un espace de Banach).

THÉORÈME 3. *Si l'opération K_A a les propriétés suivantes*:

(1) les opérations A_i sont commutatives entre elles

(2) $T_A = 0$,

(3) $SA_i = A_i S, \quad i = 0, 1, ..., n-1$

(4) le déterminant D_A admet un déterminant inverse D_A^{-1}, alors l'équation

$$Ax \equiv (A_0 + A_1 S + ... + A_{n-1} S^{n-1}) x = x_0 \tag{18}$$

a une solution

$$x = D_A^{-1} A' x_0, \tag{19}$$

où A' est déterminée par les formules (6'), (6'').

Démonstration. D'après les hypothèses et d'après le Théorème 1, nous avons

$$AA' = A'A = D_A \tag{20}$$

et nous avons une nouvelle équation, équivalente à l'équation donnée (18)

$$D_A x = A'Ax = A'x_0,$$

d'où résulte la thèse du Théorème 3.

Remarquons, qu'en vertu de l'égalité

$$SA_i = A_i S + (SA_i - A_i S) \tag{21}$$

il suffit de ne considérer que les équations involutives de la forme (1).

Nous donnerons encore les exemples des opérations commutatives avec involution. Si X est un espace des fonctions $x(t)$ intégrables sur l'intervalle $(-\infty, +\infty)$ et ayant des dérivées d'ordre arbitraire, alors la transformation de Fourier

$$Tx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)\, e^{its}\, dt = y(s)$$

est une involution d'ordre 4 [3]. En désignant

$$Mx = it\, x(t); \quad Nx = \frac{d}{ds} x(s) \tag{22}$$

nous avons, comme on le sait [1],

$$TMx = NTx; \quad TNx = -MTx. \tag{23}$$

Nous démontrerons, que chaque opération de la forme

$$C_k = M^{2k} + N^{2k} \quad (k = 1, 2, \ldots) \tag{24}$$

est commutative avec la transformation de Fourier.

En réalité, on peut facilement constater par l'induction mathématique, que

$$TM^k = N^k T, \quad TN^k = (-1)^k M^k T, \quad (k = 1, 2, \ldots) \tag{25}$$

d'où

$$T(M^{2k} + N^{2k}) = TM^{2k} + N^{2k} T + (-1)^{2k} M^{2k} T = (M^{2k} + N^{2k}) T. \tag{26}$$

Il en résulte, que les opérations de la forme (24) sont commutatives avec les opérations T, T^2, T^3.

Par analogie nous obtenons, que chaque opération M^{2k} où N^{2k} est commutative avec l'opération T^2.

Si l'opération D_1 est complète-continue, D_2 est une opération algébrique linéaire, commutative avec la transformation de Fourier, alors l'opération

$$T(D_1 + D_2) - (D_1 + D_2) T$$

est complète-continue, d'après l'égalité

$$T(D_1 + D_2) = (D_1 + D_2) T = (TD_1 - D_1 T) + (TD_2 - D_2 T) = TD_2 - D_2 T.$$

Remarquons encore, qu'on peut présenter le déterminant D_A sous la forme suivante

$$D_A = \prod_{m=0}^{n-1} d_m, \tag{27}$$

où

$$(27') \qquad d_m = \sum_{k=0}^{n-1} \varepsilon_k^m A_k \quad (\varepsilon_k = e^{\frac{2\pi}{n} ik}; \qquad k = 0, 1, \ldots, n-1).$$

La démonstration de la propriété (27) est analogue à la démonstration pour le déterminant cyclique numérique [4].

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MATHEMATICS

On the Topological Product of Paracompact Spaces

by

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Presented by K. KURATOWSKI on September 17, 1960

It is well known that the topological product of two paracompact Lindelöf spaces may fail to be a normal space. The purpose of the present note is to sketch the proofs of the following two theorems.

THEOREM 1. *The topological product of every countable indexed family of paracompact topologically complete (in the sense of E. Čech) spaces is a paracompact topologically complete space.*

THEOREM 2. *The topological product of every countable indexed family consisting of metrizable spaces and topologically complete paracompact spaces is a paracompact space.*

Let us quickly recall the definitions and basic properties of the terms which are used in the statements of the above theorems, and which will be used troughout this note.

A completely regular space P is said to be *topologically complete* (see [1]) in the sense of E. Čech, or merely *complete*, if P is a G_δ-subset of the Čech-Stone compactification $\beta(P)$ of P. If P is a complete space and $P \subset R$, $\overline{P} = R$, then P is a G_δ-subset of R. In [2] the following internal characterization of completely regular complete spaces is given: a completely regular space P is complete if and only if there exists a sequence $\{\mathfrak{A}_n\}$ of open coverings of P satisfying the following condition:

If $\mathfrak{A}$ is a family of subsets of P having the finite intersection property (i.e., $A_1, \ldots, A_k \in \mathfrak{A}$ implies $\bigcap_{i=1}^{k} A_i \neq \emptyset$) and if $\mathfrak{A} \cap \mathfrak{A}_n \neq \emptyset$ for all $n = 1, 2, \ldots$, then the intersection of closures of sets from $\mathfrak{A}$ is non-void.

Every sequence $\{\mathfrak{A}_n\}$ of open coverings satisfying the preceding condition will be called complete.

A Hausdorff space P is said to be *paracompact*, if every open covering $\mathfrak{A}$ of P has a locally-finite refinement. A Hausdorff space P is paracompact if and only if P is fully normal (see [3]), i.e., every open covering of P is normal. A normal sequence of open coverings is a sequence $\{\mathfrak{A}_n\}$ of open coverings such that $\mathfrak{A}_{n+1}$

is a star-refinement of $\mathfrak{A}_n$, i.e., for every $B \in \mathfrak{A}_{n+1}$ there exists a $C \in \mathfrak{A}_n$ such that the star

$$\bigcup \{A;\, A \in \mathfrak{A}_{n+1},\, A \cap B \neq \emptyset\}$$

of B in $\mathfrak{A}_{n+1}$ is contained in C. An open covering $\mathfrak{A}$ is normal, if there exists a normal sequence $\{\mathfrak{A}_n\}$ such that $\mathfrak{A}_1$ refines $\mathfrak{A}$. Every metrizable space is paracompact.

A mapping f of a space P onto a space Q is said to be *perfect* if f is continuous, closed (i.e., the image under f of every closed subset of P is a closed subset of Q) and the inverse images under f of points $y \in Q$ are compact subspaces of P.

LEMMA 1. *Let f be a perfect mapping of P onto Q. P is paracompact if and only if Q is paracompact. P is complete if and only if Q is complete.*

The first assertion is well known, the second may be proved using complete sequences of open coverings. Now we are prepared to prove the following

THEOREM 3. *The following two statements about a completely regular space P are equivalent:*

1° There exists a perfect mapping of P onto a complete metrizable space.

2° P is paracompact and complete.

Proof. According to the preceding Lemma the condition 1° implies the condition 2°. Conversely, let us suppose 2°. The space P being complete, there exists a complete sequence $\{\mathfrak{A}_n\}$ of open coverings of P. The space P being paracompact, every $\mathfrak{A}_n$ is an open normal covering. It is easy to construct a normal sequence $\{\mathfrak{B}_n\}$ of open coverings such that every $\mathfrak{B}_n$ refines $\mathfrak{A}_n$. It follows at once that the sequence $\{\mathfrak{B}_n\}$ is complete. There exists a pseudometric φ on P such that

(a) for every $n = 1, 2, \ldots$, there exists a positive real number ε such that the covering consisting of all open spheres of radius ε refines $\mathfrak{A}_n$, and

(b) for every positive real number ε there exists an n such that $\mathfrak{A}_n$ refines the covering consisting of all open φ-spheres of radius ε.

Finally, there exists a mapping f of P onto a metric space (Q, ψ) such that

$$\varphi(x, y) = \psi(f(x), f(y))$$

for all x and y in P. Clearly, f is a continuous mapping. To prove that f is a closed mapping it is sufficient to show that whenever y is an accumulation point of $f[M]$, then there exists an accumulation point x of M, $x \in f^{-1}[y]$. Let $\mathfrak{W}$ be a maximal family of subsets of P having the finite intersection property and containing all sets of the form $M \cap f^{-1}[U]$, when U runs over all neighbourhoods of y. According to (a), we have $\mathfrak{W} \cap \mathfrak{B}_n \neq \emptyset$ for all n. Thus, the intersection of all $\bar{M}$, $M \in \mathfrak{W}$, is non-void. Evidently, this intersection is contained in $f^{-1}[y]$. To prove compactness of $f^{-1}[y]$, where y is a point of Q, it is sufficient to note that, for every maximal family of subsets of P having the finite intersection property and containing the set $f^{-1}[y]$, we have by (a)

$$\mathfrak{W} \cap \mathfrak{B}_n \neq \emptyset, \quad n = 1, 2, \ldots.$$

Thus, f is a perfect mapping of P onto Q. It is easy to see that (Q, ψ) is a complete metric space.

To prove the Theorems 1 and 2 we shall need the following simple

LEMMA 2. *The Cartesian product of perfect mappings is a perfect mapping.*

Of course, the Cartesian product of mappings f_a of P_a to Q_a is the mapping f of the Cartesian product of P_a to the Cartesian product of Q_a defined as follows:

$$f(\{x_a;\ a \in A\}) = \{f_a(x_a);\ a \in A\}.$$

The preceding Lemma is a consequence of the following

LEMMA 3. *Let us suppose that U is an open subset of the topological product*

$$P = X\{P_a;\ a \in A\}$$

containing a compact set K of the form

$$K = X\{K_a;\ a \in A\},$$

where $K_a \subset P_a$.

There exists an open set $V \subset P$ of the form

$$V = X\{V_a;\ a \in A\}$$

with $K \subset V \subset U$.

The proof of Lemma 3 is quite standard and may be left to the reader.

A class T of topological spaces is said to be *countably productive*, if the topological product of every countable indexed family consisting of spaces from T belongs to T. From Lemma 2 we have at once the following

LEMMA 4. *Let T be a countably productive class of topological spaces. The class of all inverse images under perfect mappings of spaces from T is countably productive.*

The Theorems 1 and 2 are immediate consequences of the preceding Lemma and of Theorem 3.

According to Theorem 3, every complete paracompact space is the inverse image under a perfect mapping of a complete metrizable space. It is not true that every paracompact space is the inverse image, under a perfect mapping, of a metrizable space. Indeed, the family of all such spaces is countably productive and the topological product of two paracompact spaces may fail to be a paracompact space. We shall prove the following

THEOREM 4. *The following statements about a completely regular space P are equivalent:*

1° P is paracompact.

2° If R is a complete completely regular space containing P as a dense subspace, then there exists a paracompact topologically complete space S with $P \subset S \subset R$.

3° If R is a locally compact completely regular space containing P as a dense subspace, then there exists a paracompact topologically complete space S with $P \subset S \subset R$.

Proof. Let us suppose 1° and let R be a space from 2°. Let $\{\mathfrak{A}_n\}$ be a complete sequence of open coverings of R. The space P being paracompact, we can choose a normal sequence $\{\mathfrak{B}_n\}$ of open coverings of P such that every $\mathfrak{B}_n$ refines $\mathfrak{A}_n$.

Let U_n be the union of all open $U \subset R$ for which $U \cap P \in \mathfrak{B}_n$. Put

$$S = \bigcap \{U_n; \quad n = 1, 2, \ldots\}.$$

Let us denote by $\mathfrak{C}_n$ the family of all open subsets U of S for which $U \cap P \in \mathfrak{B}_n$. Evidently, $\{\mathfrak{C}_n\}$ is a normal sequence of open coverings of S. We shall prove that the sequence $\{\mathfrak{C}_n\}$ is complete. Let $\mathfrak{W}$ be a maximal family of subsets of S with the finite intersection property such that $\mathfrak{C}_n \cap \mathfrak{W} \neq \varnothing$ for all n. Let $\mathfrak{N}$ be a maximal family of subsets of R with the finite intersection property and $\mathfrak{N} \supset \mathfrak{W}$. Clearly, $\mathfrak{N} \cap \mathfrak{A}_n \neq \varnothing$ for all n. Thus,

$$F = \cap \{\overline{N}^R; \quad N \in \mathfrak{N}\} \neq \varnothing.$$

It is sufficient to prove that $F \subset S$. If $C \in \mathfrak{W} \cap \mathfrak{C}_{n+1}$, then the star of C in $\mathfrak{C}_{n+1}$ is contained in some $C_1 \in \mathfrak{C}_n$. It follows that the closure of C is contained in C_1. But $C_1 \subset U_n$; thus, $F \subset \bigcap_{n=1}^{\infty} U_n = S$. Now let φ be the pseudometric from the proof of the Theorem 3 and let (Q, ψ) be the corresponding metric space. The mapping of S onto (Q, ψ) is perfect and (Q, ψ) is a complete metric space. By Theorem 3 the space S has the required properties.

Evidently, 2° implies 3°. It remains to prove that 3° implies 1°. Let K be a compactification of P and let $\mathfrak{A}$ be an open covering of P. Let $\mathfrak{B}$ be the family of all open subsets U of K with $U \cap P \in \mathfrak{A}$. The union of the family $\mathfrak{B}$ is a locally compact subspace R of K. Let S be the space from 3°. There exists a locally finite open refinement $\mathfrak{C}$ of the family of all $B \cap S$, $B \in \mathfrak{B}$. Evidently, the family of all $C \cap P$, $C \in \mathfrak{C}$, is a locally finite refinement of $\mathfrak{A}$. The proof is complete.

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MATHÉMATIQUE

Sur la continuité régulière des dérivées d'ordre maximum des intégrales généralisées de Poisson-Weierstrass relatives au système parabolique

par

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Présenté par T. WAŻEWSKI le 27 septembre 1960

Soit le système de $N \geqslant 1$ équations aux dérivées partielles d'ordre $M \geqslant 2$ de la forme

$$(1) \qquad \sum_{1 \leqslant j \leqslant N}^{0 \leqslant k_1 + \ldots + k_n \leqslant M} A_{\alpha j}^{k_1 \ldots k_n}(X, t) \frac{\partial^{k_1 + \ldots + k_n} u_j}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} - \frac{\partial u_\alpha}{\partial t} = 0, \quad \alpha = 1, 2, \ldots, N$$

aux fonctions inconnues $u_1(X, t), \ldots, u_N(X, t)$. Les coefficients $A_{\alpha j}^{k_1 \ldots k_n}(X, t)$ sont des fonctions complexes, définies, continues et bornées dans la région

$$(2) \qquad X(x_1, \ldots, x_n) \in E; \quad 0 \leqslant t \leqslant T.$$

E étant un espace euclidien à n dimensions.

Nous admettons que ces coefficients sont hölderiens par rapport à la variable spatiale X, à l'exposant h; on admet en outre que les coefficients des dérivées de l'ordre le plus élevé M sont hölderiens par rapport à la variable t, à l'exposant h'. Conformément à la définition du système parabolique, donnée par Petrovsky [1], nous admettons que l'équation

$$(3) \qquad \det_{\alpha, j} \left| \sum_{k_1 + \ldots + k_n = M} A_{\alpha j}^{k_1 \ldots k_n}(X, t)(is_1)^{k_1} \ldots (i s_n)^{k_n} - \delta_\alpha^j \lambda \right| = 0$$

a les racines en λ, dont toutes les parties réelles sont inférieures à un nombre négatif fixé-δ

$$(4) \qquad \mathrm{Re}(\lambda) < -\delta < 0$$

pour toutes les valeurs des variables réelles $s_1, \ldots, s_n$ vérifiant l'égalité

$$(5) \qquad s_1^2 + s_2^2 + \ldots + s_n^2 = 1$$

et pour tout point de la région

$$X \in E; \quad 0 \leqslant t \leqslant T.$$

Dans [2] nous avons construit, sous les hypothèses précédentes, une matrice $\{\Gamma_{\alpha\beta}(X, t;\ Y, \tau)\}$ des solutions fondamentales pour le système (1). Dans [4] nous avons étudié les intégrales généralisées de Poisson-Weierstrass

$$J_\alpha(X, t) = \int\int_E\int \sum_{\beta=1}^{N} \Gamma_{\alpha\beta}(X, t;\ Y, 0) f_\beta(Y)\, dY, \qquad (\alpha = 1, 2, ..., N) \tag{6}$$

et leurs dérivées spatiales jusqu'à l'ordre $M - 1$.

Le travail présent constitue une étude de la continuité des dérivées de l'ordre le plus élévé M des fonctions (6), qui exige des considérations plus délicates. Nous admettons pour les fonctions complexes données $f_\beta(Y)$ les hypothèses suivantes (un peu plus générales que dans [4]).

1) Les fonctions $f_\beta(Y)$ sont définies dans l'espace E entier.

2) Les fonctions $f_\beta(Y)$ sont intégrables dans tout domaine mesurable et borné, situé dans E.

3) Les fonctions $f_\beta(Y)$ vérifient en tout point Y de l'espace E l'inégalité

$$|f_\beta(Y)| < a \exp[b\,|Y X_0|^\chi], \tag{7}$$

a et b étant des constantes positives, $|Y X_0|$ désigne la distance entre le point Y et un point X_0 fixé dans l'espace E, χ est une constante positive vérifiant les inégalités $1 \leqslant \chi < M(M-1)^{-1}$.

Nous allons démontrer le théorème suivant, exprimant une certaine régularité de la continuité des dérivées d'ordre M des intégrales (6).

Théorème. *Si les fonctions données $f_\beta(Y)$ vérifient les conditions* 1), 2), 3), *alors les dérivées spatiales d'ordre M des intégrales généralisées de Poisson-Weierstrass* (6) *vérifient en tout l'espace E pour $0 < t < T$ l'inégalité*

$$|D_x^{(M)}[J_\alpha(X, t)]| < C t^{-\mu} \exp[2^{\chi-1} b\,|X X_0|^\chi] \tag{8}$$

et la suivante condition généralisée de Hölder

$$|D_X^{(M)}[J_\alpha(X, t)] - D_X^{(M)}[J_\alpha(X_1, t_1)]| < \tag{9}$$

$$< \frac{C' \exp[2^{\chi-1} b\,|X X_0^\chi|]}{t^\mu} [|X X_1|^{\theta h_1} + |t - t_1|^{\theta' h/(h+M)}],$$

où $|X X_0| \geqslant |X_1 X_0|$, $0 < t < t_1 < T$, $h_1 = \min(h, 2h')$, μ est une constante arbitrairement supérieure à $1 - M^{-1}$, θ et θ' sont deux constantes positives arbitrairement inférieures à unité, la constante positive C dépend du choix de μ et la constante positive C' — du choix des μ, θ, θ'.

Démonstration. Nous donnerons la preuve abrégée du théorème en nous appuyant principalement sur les résultats présentés dans nos travaux [3] et [4]. D'après ces travaux, nous pouvons écrire les intégrales étudiées (6) sous la forme d'une somme

$$J_\alpha(X, t) = \overset{*}{J}_\alpha(X, t) + \overset{**}{J}_\alpha(X, t) \tag{10}$$

de l'intégrale

$$(11) \qquad \overset{*}{J}_{\alpha}(X,t) = \int\int\int_{E} \sum_{\beta=1}^{N} W_{\alpha\beta}^{Y,0}(X,t;\ Y,0) f_{\beta}(Y)\, dY$$

et du quasi-potentiel de charge spatiale

$$(12) \qquad \overset{**}{J}_{\alpha}(X,t) = \int_{0}^{t} \int\int_{E} \sum_{\beta=1}^{N} W_{\alpha\beta}^{\Pi,\xi}(X,t;\ \Pi,\xi)\, \varrho_{\beta}(\Pi,\xi)\, d\Pi d\xi$$

de densité $\{\varrho_\beta\}$ déterminée par les intégrales

$$(13) \qquad \varrho_{\beta}(\Pi,\xi) = \int\int\int_{E} \sum_{\gamma=1}^{N} \Phi_{\beta\gamma}(\Pi,\xi;\ Y,0) f_{\gamma}(Y)\, dY.$$

$W_{\alpha\beta}^{\Pi\xi}$ sont des éléments d'une matrice, dite de quasi-solutions et les fonctions $\Phi_{\beta\gamma}$ sont des solutions d'un système d'équations de Volterra (voir [2], [4]). D'après ces travaux, les dérivées spatiales d'ordre quelconque m des éléments de quasi-solutions vérifient les inégalités

$$(14) \qquad |D_X^{(m)}[W_{\alpha\beta}^{z,\zeta}(X,t;\ Y,\tau)]| < \frac{C_m}{(t-\tau)^{(m+n)/M}} \exp\left[-\frac{c_m |XY|^{M/(M-1)}}{(t-\tau)^{1/(M-1)}}\right]$$

d'où résultent les limitations

$$(15) \qquad |D_X^{(m)}[W_{\alpha\beta}^{z,\zeta}(X,t;\ Y,\tau)]| \begin{cases} < \dfrac{\text{const}\ (t-\tau)^{-\mu'}}{|XY|^{n+m-M\mu'}}, & \text{si } |XY| \leqslant \varrho_0 \\ < \text{const}\ (t-\tau)^{\mu''} e^{-k|XY|^{\chi}}, & \text{si } |XY| > \varrho_0 \end{cases}$$

la constante χ est fixée arbitrairement dans l'intervalle $1 \leqslant \chi < M/(M-1)$, la constante μ' est fixée arbitrairement dans l'intervalle $mM^{-1} < \mu' < (m+n)\, M^{-1}$, les constantes positives ϱ_0, μ'', k peuvent être fixées arbitrairement. Il en résulte que les fonctions (11) admettent, pour $0 < t < T$, les dérivées de tout ordre m présentées par les intégrales absolument convergentes

$$(16) \qquad D_X^{(m)}[\overset{*}{J}_{\alpha}(X,t)] = \int\int\int_{E} \sum_{\beta=1}^{N} D_X^{(m)}[W_{\alpha\beta}^{Y,0}(X,t;\ Y,0)] f_{\beta}(Y)\, dY,$$

nous en concluons l'inégalité ($k > 2^{\chi-1} b$)

$$(17) \qquad |D_X^{(m)}[\overset{*}{J}_{\alpha}(X,t)]| < \text{const} \int\int\int_{K_{\rho_0}} \frac{a \exp[b|YX_0|^{\chi}]\, dY}{t^{\mu'} |XY|^{n+m-M\mu'}} +$$

$$+ \exp[2^{\chi-1} b |XX_0|^{\chi}]\, t^{\mu''} \int\int\int_{E-K_{\rho_0}} \exp[-(k - 2^{\chi-1} b)|XY|^{\chi}]\, dY <$$

$$< \text{const}\ t^{-\mu} \exp[2^{\chi-1} b |XX_0|^{\chi}],$$

K_{ρ_0} étant une sphère centrée en X, de rayon fixé ϱ_0, μ est une constante positive fixée arbitrairement au dessus de m/M. En s'appuyant sur l'inégalité (17) pour $m = M+1$, nous déduisons l'inégalité généralisée de Hölder

$$(18)\quad |D_X^{(M)}[\overset{*}{J}_\alpha(X,t)] - D_X^{(M)}[\overset{*}{J}_{\alpha'}(X_1,t)]| < \text{const}\, t^{-\mu} \exp[2^{\varkappa-1} b\, |XX_0|^\varkappa]\, |XX_1|,$$

X et X_1 étant les deux points arbitraires de l'espace E; on a $|XX_0| \geqslant |X_1 X_0|$, $\mu > 1 + M^{-1}$, $0 < t < T$. En posant $m = 2M$ dans (17), on aura

$$(19)\quad |D_{Xt}^{(M+1)}[\overset{*}{J}_\alpha X, t)]| < \text{const}\, t^{-\bar{\mu}} \exp[2^{\varkappa-1} b\, |XX_0|^\varkappa]$$

où $\bar{\mu} > 2$, d'où il résulte

$$(20)\quad |D_X^{(M)}[\overset{*}{J}_\alpha(X,t)] - D_X^{(M)}[\overset{*}{J}_\alpha(X,t_1)]| < \frac{\text{const}}{t^{\mu_0+\mu_0'}} \exp[2^{\varkappa-1} b\, |XX_0{}^\varkappa|]\, |t-t_1|^{\mu_0},$$

μ_0, μ_0' sont des constantes positives arbitrairement chosies, de manière que $0 < \mu_0 < 1$, $\mu_0' > 1$; $0 < t < t_1 < T$.

Pour étudier la seconde composante (12), remarquons que les fonctions (13), en vertu des travaux [3], [4], vérifient les inégalités

$$(21)\quad |\varrho_\beta(\Pi, \xi)| < \frac{\text{const}}{\xi^{\mu_1}} \exp[2^{\varkappa-1} b\, |\Pi X_0|^\varkappa],$$

$$(22)\quad |\varrho_\beta(\Pi, \xi) - \varrho_\beta(\Pi_1, \xi)| < \frac{\text{const}}{\xi^{\mu_1}} \exp[2^{\varkappa-1} b\, |\Pi X_0|^\varkappa]\, |\Pi\Pi_1|^{\theta h_1}$$

où $1 - h_1 M^{-1} < \mu_1 < 1$, $0 < \theta < 1$, $h_1 = \min(h, 2h')$, $|\Pi X_0| \geqslant |\Pi_1 X_0|$.

Soit une sphère K, de rayon 2, centrée au point arbitraire X de l'espace E. Décomposons l'intégrale (12) en une somme d'intégrales

$$(23)\quad \overset{**}{J}_\alpha(X,t) = \overset{**}{J}_\alpha^{(K)}(X,t) + \overset{**}{J}_\alpha^{(E-K)}(X,t)$$

étendues à la sphère K et à la partie extérieure $E - K$. X étant point arbitraire de K dont la distance du centre $\bar{X}$ est inférieure à l'unité. D'après nos études (voir [3], formule (64)), en tenant compte des propriétés (21) et (22), nous concluons que la première composante de la somme (23) possède les dérivées spatiales d'ordre M, pour $|XX| < 1$, vérifiant l'inégalité

$$(24)\quad D_X^{(M)}[\overset{**}{J}_\alpha^{(K)}(X,t)] < \text{const} \int_0^t \frac{d\xi}{(t-\xi)^{\mu_*} \xi^{\mu_1}} \exp[2^{\varkappa-1} b\, XX_0{}^\varkappa],$$

où $1 - \theta h_1 M^{-1} < \mu_* < 1$. Ensuite, en vertu du Théorème 8 du travail [3], nous aurons l'inégalité généralisée de Hölder de la forme

$$(25)\quad |D_X^{(M)}[\overset{**}{J}_\alpha^{(K)}(X,t)] - D_X^{(M)}[\overset{**}{J}_\alpha^{(K)}(X_1,t_1)]| < \\ < \text{const}\, t^{-\mu} \exp[2^{\varkappa-1} b\, |XX_0|^\varkappa]\, [|XX_1|^{\theta h_1} + |t-t_1|^{\theta'' h/(h+M)}].$$

θ, θ'' étant deux constantes positives, arbitrairement inférieures à l'unité, on a de même $|X_1 X| < 1$, $0 < t < t_1 < T$. Quant à la seconde composante de la somme (23),

elle est bornée, ainsi que toutes ses dérivées spatiales, si $t \to 0$, puisque $|X\Pi| > 1$, si $\Pi \in E - K$; nous aurons donc

$$(26) \quad |D_X^{(M)}[\overset{**}{J}^{(E-K)}(X,t)] - D_X^{(M)}[\overset{**}{J}^{(E-K)}(X_1,t_1)]| < \\ < \text{const} \exp[2^{\chi-1} b |XX_0|^{\chi}][|XX_1| + |t - t_1|].$$

Nous avons supposé jusqu'ici, que les points X et X_1 se trouvent à une distance du centre $\overline{X}$ inférieure à l'unité. Or, le choix du centre $\overline{X}$ de la sphère K était arbitraire.

Si l'on a $|XX_1| \geqslant 2$, la propriété (25), concernant la distance $|XX_1|$, résulte immédiatement de l'inégalité (24). Nous concluons donc la propriété de Hölder de la forme (25) aussi pour la somme (23), quelques soient les points X, X_1 de l'espace E. En rapprochant les résultats obtenus (17), (18), (20), (24), (25), (26), on arrive aux conclusions (8) et (9) de notre Théorème.

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MATHEMATICS

Some Remarks on Homeomorphism of Banach Spaces

by
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Presented by S. MAZUR on October 27, 1960

1.

The problem posed by Banach: "Are all infinite-dimensional separable B-spaces homeomorphic?" still remains unsolved. However, partial results obtained by several authors (see [12], [8], [9], [11]) suggest a positive answer to this question. The most important results are:

(K_1) *The spaces c_0 and l are homeomorphic, where c_0 denotes the space of all real sequences converging to zero and l — the space of all absolutely convergent real series* (Kadeč [8]).

(K_2) *Every infinite-dimensional separable reflexive B-space is homeomorphic to l* (Kadeč [9]).

(Kl) *Every infinite-dimensional separable conjugate B-space is homeomorphic to l* (Klee [11] and Kadeč [16]).

In this paper the following main result is presented:

(B—P) *Every separable B-space which contains a closed subspace which is either homeomorphic to l or may be linearly mapped onto a B-space homeomorphic to l, is homeomorphic to l.*

It follows from the results listed above that all known special infinite-dimensional separable B-spaces (for instance all special spaces investigated by Day [5] or Dunford and Schwartz [6]) are homeomorphic. In particular, we have:

COROLLARY 1. *If Q is an infinite compact metric space then the space $C(Q)$ — of all continuous real functions defined on Q — is homeomorphic to the space l* *).

The proof of (B—P) is based on the following result, due to Bartle and Graves [2], (see also [13]):

(*) *Let E be a closed subspace of a B-space X, Φ — the natural homeomorphism of X onto the quotient space X/E. Then there is a continuous mapping φ (a selection) of X/E into X such that $\varphi[x] \in \Phi^{-1}[x]$ for every $[x] \in X/E$* **).

*)In particular we obtain the result of Kadeč and Levin [10] which states that the spaces $C(J)$ and $C(J \times J)$, where $J = \langle 0,1 \rangle$, are homeomorphic.

**) We are indebted to Professors V. Klee and E. Michael, who called our attention to this result.

2. Notations

We intend to preserve the terminology and notations adopted in [6]. In particular, the symbols l, c_0, $C(Q)$, etc. have the same meaning as in [6], Chap. IV. We shall write $X \sim Y$, if the space X and Y are homeomorphic and $X = Y$, if they are isomorphic (linearly homeomorphic). The symbol $X \times Y$ will denote the Cartesian product of spaces X and Y. Let (X_i) be the sequence of B-spaces with norms $\|\cdot\|_i$, respectively $(i = 1, 2, \ldots)$. The symbol $(X_1 \times X_2 \times \ldots)_{c_0}$ will denote the Banach space whose elements are sequences (x_i) with $x_i \in X_i$ $(i = 1, 2, \ldots)$ such that $\lim_{i \to \infty} \|x_i\|_i = 0$, under the norm $\|(x_i)\| = \sup_i \|x_i\|_i$. The term "subspace" of X will always mean a closed linear manifold in X.

3. Lemmas

LEMMA 1. *If X is a B-space and E — a subspace of X, then $X \sim X/E \times X$* (Klee [11]).

Proof. The required homeomorphic mapping ψ may be given by the formula $\psi([x], e) = \varphi[x] + e$ for $([x], e) \in (X/E) \times E$, where φ has the same meaning as in (*).

LEMMA 2. *If B-spaces X and Y are homeomorphic, then $(X \times X \times \ldots)_{c_0} \sim (Y \times Y \times \ldots)_{c_0}$.*

Proof. Let T be a homeomorphism of X onto Y. Without loss of generality we may assume that $T0 = 0$. Put $S(x_i) = (Tx_i)$. It can easily be seen that S is a homeomorphism of $(X \times X \times \ldots)_{c_0}$ onto $(Y \times Y \times \ldots)_{c_0}$.

LEMMA 3. *Every infinite-dimensional subspace E of the space l is homeomorphic to l.*

Proof. According to lemma 2 in [14], p. 214, the subspace E contains a complemented subspace E_1, isomorphic to l. Thus, we have $E = E/E_1 \times l$. Since $l = l \times l$, we have

$$E = E/E_1 \times l = E/E_1 \times (l \times l) = E \times l. \tag{1}$$

On the other hand, according to (K_1), Lemma 2, Lemma 1 and to the fact that $(c_0 \times c_0 \times \ldots)_{c_0} = c_0$ (see [14], p. 212, Proposition 3), we have

$$l \sim c_0 = (c_0 \times c_0 \times \ldots)_{c_0} \sim (l \times l \times \ldots)_{c_0} \sim ((E \times l/E) \times (E \times l/E) \times \ldots)_{c_0}.$$

Thus, using the properties of "Cartesian product in the sense c_0" ([14], p. 212, Proposition 3), we obtain

$$l \sim ((E \times l/E) \times (E \times l/E) \times \ldots)_{c_0} = (E \times E \times \ldots)_{c_0} \times (l/E \times l/E \times \ldots)_{c_0} = \tag{2}$$
$$= E \times ((E \times l/E) \times (E \times l/E) \times \ldots)_{c_0} \sim E \times l.$$

From (1) and (2) it follows immediately that $E \sim l$.

Remark 1. Similarly one may prove that if Y is a subspace of the space $l(\aleph)$*) and $Y \sim X \times l(\aleph)$, then $Y \sim l(\aleph)$.

*) The symbols $l(\aleph)$, $c_0(\aleph)$, $m(\aleph)$ have the same meaning as the symbols $l(S)$, $c_0(S)$, $m(S)$, for $\overline{\overline{S}} = \aleph$ in Day's monography [5], p. 29.

4. Proof of basic results

PROPOSITION 1. *Let X be a separable B-space. Then the following conditions are equivalent*:

(a) *X is homeomorphic to the space l;*

(b) *the space X contains a subspace E homeomorphic to l;*

(c) *there exists a metric space A such that $X \sim A \times l$;*

(d) *the spaces X and $X \times l$ are homeomorphic.*

Proof. (a) $\rightarrow$ (b) is trivial. (b) $\rightarrow$ (c) is an immediate consequence of Lemma 1. (c) $\rightarrow$ (d) follows from the remark that, since $l = l \times l$, we have $X \sim A \times l \sim A \times (l \times l) = (A \times l) \times l \sim X \times l$.

(d) $\rightarrow$ (a). According to a lemma of Schauder (see [1] p. 111) there exists a linear mapping U from l onto X. Let $E = U^{-1}0$. Since $l = l \times l$, we may assume without loss of generality that E is an infinite dimensional subspace of the space l. Since $X = l/E$, we obtain $l \sim E \times l/E = E \times X \sim l \times X$ by applying Lemmas 1 and 3. Thus, (d) implies (a).

Proof of (B—P). Let E be a subspace of a separable B-space X. If $E \sim l$, then, according to Proposition 1 (b), $X \sim l$. If there exists a linear mapping U of E onto a space Y homeomorphic to l, then, according to Lemma 1, we have $E \sim E/F \times F = Y \times F \sim l \times F$, where $F = U^{-1}0$. According to Proposition 1 (d), we then have $E \sim l$, whence $X \sim l$.

Remark 2. The Proposition 1 and the result (B—P) remain valid if we replace the assumption "X is separable" by the assumption "density character of X is equal $\aleph$" and we replace everywhere the space l by the space $l(\aleph)$.

Proof of Corollary 1. The Corollary follows immediately from the well known fact that if Q is an infinite compact metric space, then the space $C(Q)$ contains a subspace E isomorphic to c_0. Thus, by (K_1) $E \sim l$ and by Proposition 1, $C(Q)$ is homeomorphic to l.

COROLLARY 2. *If Q is a countable compact metric space, then every infinite-dimensional subspace X of the space $C(Q)$ is homeomorphic to l.*

This follows from (K_1), (B—P) and the fact that X contains a subspace isomorphic to c_0 ([15], p. 214).

COROLLARY 3. *Every infinite-dimensional subspace of a B-space with an unconditional basis is homeomorphic to l.*

Proof. Let X be an infinite-dimensional subspace of a space with an unconditional basis. If X is reflexive, then $X \sim l$, by (K_2). In the other case, according to a generalization of the result of R. C. James ([7] and [4]), X contains either a subspace isomorphic to l or a subspace isomorphic to c_0. In both cases the condition (b) of Proposition 1 is satisfied.

COROLLARY 4. *Let Q be a compact metric space, μ — a Borel measure on Q. Then every infinite dimensional subspace B of the space $L(Q, \mu)$ — of all real-valued μ-absolutely summable functions on Q — is homeomorphic to l.*

This follows from the fact that every non-reflexive subspace of the space $L(Q, \mu)$ contains a subspace isomorphic to l *).

COROLLARY 5. *If* $\aleph > \aleph_0$, *then* $m(\aleph) \sim l(2^{\aleph})$.

This follows immediately from Remark 2 and from the fact that the space $m(\aleph)$ contains a subspace isometrically isomorphic to $l(2^{\aleph})$.

5. Final remarks

It follows from the preceding considerations that if there would exist an infinite-dimensional separable B-space X non-homeomorphic to l, then this space would have the following strange linearly topological properties:

(1) No infinite-dimensional subspace of X has an unconditional basis (cf. [4], p. 173.5.1).

(2) No infinite-dimensional subspace of X is isomorphic to a conjugate space.

(3) Every reflexive subspace of the space X^*, conjugate to X, is finite-dimensional **).

(4) No subspace of X is isomorphic to c_0 ***).

(5) If Y is a subspace of X then the space $X_1 = X/Y$ has the properties (1)—(4).

The existence of such space seems very improbable.

To show that every infinite-dimensional separable B-space has the property (a) of Proposition 1 it suffices to establish that every such a space contains a subspace having the so called T-system with Bernstein property, [8].

It would be interesting to generalize Corollary 1 to the case of spaces $C(Q)$ for arbitrary compact Hausdorff spaces Q, i.e. to show that the space $C(Q)$ is homeomorphic to a suitable space $l(\aleph)$. We do not know, whether the spaces $c_0(\aleph)$ and $l(\aleph)$ are homeomorphic for $\aleph > \aleph_0$.

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*) This follows from (B—P) and a remark of V. Ptak (unpublished) that if the space X^* contains a reflexive subspace Y^*, then there exists a linear mapping from X onto Y.

**) This fact is due to M. J. Kadeč and one of the authors. The proof will appear in Studia Mathematica.

***) This follows from (B—P) and Theorem 4 in [3].

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MATHEMATICS

Algebras with a Quasi-involution

by

B. GLEICHGEWICHT

Presented by E. MARCZEWSKI on October 31, 1960

1. Let P be an associative and commutative ring with a unit element. An operation * defined on the ring P is called an involution, if it satisfies the following conditions

$$(\alpha+\beta)^* = \alpha^*+\beta^*, \quad (\alpha^*)^* = \alpha, \quad (\alpha\beta)^* = \alpha^*\beta^*$$

for any $\alpha, \beta \in P$. The aim of this note is to study a class of algebras over the ring P with an involution.

A set A is said to be an *algebra with quasi-involution* over the ring P, if

1° A is a modul over P,

2° A is closed with respect to a product xy such that the two-sided distributive law holds and

(i) $$(\alpha x)\, y = \alpha\,(xy), \quad x\,(\alpha y) = \alpha^*\,(xy)$$

are true for any $\alpha \in P$ and $x, y \in A$. We remark that the product is not necessarily associative.

3° An operation * is defined on A and satisfies the conditions

(ii) $$(x+y)^+ = x^+ + y^+, \quad (\alpha x)^+ = \alpha^*\, x^+, \quad (x^+)^+ = x,$$

and

(iii) $$(xy)^+ = yx, \quad x\,(yz) = (xz^+)\, y$$

for any $\alpha \in P$ and $x, y, z \in A$. The operation $^+$ will be called a *quasi-involution*.

2. Let A_0 be an associative algebra with an involution * over P satisfying in addition to the usual requirements the condition $(\alpha x)^* = \alpha^* x^*$ for any $\alpha \in P$ and $x \in A_0$, [4]. In the sequel we shall denote by $\mathcal{K}(A_0)$ the set A_0 with usual addition, multiplication by elements of P and multiplication defined as follows

$$xy = y^* \bigcirc x,$$

where $\bigcirc$ is the product operation in A_0.

If A_0 is the algebra of square matrices of fixed order, then $\mathcal{K}(A_0)$ coincides with the algebra of cracovians introduced by T. Banachiewicz, [1]. Therefore, for arbitrary

A_0, $\mathcal{K}(A_0)$ will be called a *cracovian algebra* generated by A_0. The theory of cracovian rings is developed in [2] and [3].

THEOREM 1. *If A_0 is an associative algebra with an involution, then $\mathcal{K}(A_0)$ is an algebra with quasi-involution, where the involution in A_0 is taken as the quasi-involution in $\mathcal{K}(A_0)$.*

To prove our Theorem it is sufficient to show that in $\mathcal{K}(A_0)$ the two-sided distributive law and equalities (i) and (iii) are true.

THEOREM 2. *Every algebra with a quasi-involution is equal to the cracovian algebra generated by an associative algebra with an involution.*

Let A be an algebra with a quasi-involution over a ring P. By A_0 we denote the set A with usual addition, multiplication by elements of P, where the involution and $\bigcirc$-multiplication are defined by formulas $x^* = x^+$ and

$$x \bigcirc y = yx^+. \tag{1}$$

First we prove that A_0 is an associative algebra with an involution. Using (1) we get two-sided distributive and associative laws. From (i) and (ii) we obtain the equalities $(\alpha x) \bigcirc y = \alpha(x \bigcirc y)$ and $x \bigcirc (\alpha y) = \alpha(x \bigcirc y)$. To prove that the operation* is an involution it suffices to show that $(x \bigcirc y)^* = y^* \bigcirc x^*$ what we get by using (iii). Thus, A_0 is an associative algebra with an involution. Finally, from the equality $y^* \bigcirc x = xy$ follows $A = \mathcal{K}(A_0)$.

3. In [2] the notion of τ-rings was introduced. An analogous notion can be introduced for algebras. Namely, a set B is said to be a *τ-algebra* over a ring P with an involution if

1° B is a modul over P,

2° B is closed with respect to a product xy such that the two-sided distributive law holds and $(\alpha x)\, y = \alpha\,(xy)$, $x\,(\alpha y) = \alpha^*\,(xy)$ are true for any $\alpha \in P$ and $x, y \in B$.

3° There exists an element $\tau \in B$ such that for every $x, y, z \in B$ the following equalities hold

$$x\,(yz) = (x\,(\tau z))\, y, \qquad \tau\,(\tau x) = x. \tag{iv}$$

THEOREM 3. *Every τ-algebra is an algebra with quasi-involution defined by means of formula $x^+ = \tau x$.*

THEOREM 4. *An algebra with a quasi-involution is a τ-algebra with $x^+ = \tau x$ if and only if it is the cracovian algebra generated by an associative algebra with a unit element.*

If we suppose that the generating associative algebra A_0 has an unit element e we prove that in algebra $A = \mathcal{K}(A_0)$ the element e can be taken as an element τ satisfying (iv). Let A be a τ-algebra. By Theorem 2 it can be represented by a cracovian algebra $A = \mathcal{K}(A_0)$, where A_0 is an associative algebra and the multiplication in A_0 is given by the formula $x \bigcirc y = y\,(\tau x)$. Then we prove that τ is the unit element of A_0.

THEOREM 5. *Every algebra with a quasi-involution can be embedded in a τ-algebra.*

By Theorem 2 an algebra A with a quasi-involution is the cracovian algebra generated by an associative algebra A_0 with involution: $A = \mathcal{K}(A_0)$. Since the

algebra A_0 is contained in an associative algebra A_0 with a unit element, we have $A \subset \mathcal{K}(A_0)$. But, according to Theorem 4, $\mathcal{K}(A_0)$ is a τ-algebra, which completes the proof our Theorem.

Remark. As a τ-algebra containing the algebra A with a quasi-involution the algebra of all pairs $\langle \alpha, x \rangle$ ($\alpha \in P$, $x \in A$) can be taken. The operations in this algebra are defined as follows

$$\langle \alpha, x \rangle + \langle \beta, y \rangle = \langle \alpha+\beta, x+y \rangle, \quad \alpha \langle \beta, x \rangle = \langle \alpha\beta, \alpha x \rangle,$$

$$\langle \alpha, x \rangle \langle \beta, y \rangle = \langle \alpha\beta^*, \alpha y^+ + \beta^* x + xy \rangle.$$

It is very easy to verify that the element $\tau = \langle 1,0 \rangle$ (1 is the unit element of P) satisfies conditions (iv). The isomorphic embedding of A into the considered algebra is given by formula $x \to \langle 0, x \rangle$.

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Approximate Formulas for the Diffracted Electromagnetic Wave. II

by

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Presented by W. RUBINOWICZ on August 6, 1960

When computing the intensity of a plane electromagnetic wave, diffracted on the edge of a half-plane near the boundary of geometrical shadow from Kottler's theory, we see that the formulas obtained are in agreement with Sommerfeld's strict solution in the same region.

This would provide ground for presuming that the stationary phase method, if applied to formulas (1.1) of [1], should yield reasonable results in the mentioned region. Senior's [2] theory provides still better ground for this, as will be seen when comparing the present author's formulas (3.5) in [1] with the respective formulas of Senior ("I" currently denotes [1]). Our aim is, however, to derive formulas not restricted to the case of the half-plane.

As a first step in this direction, let us verify whether the two operations of differentiation and of the stationary phase method are commutative. We consider the integral

$$J = \frac{\partial}{\partial x_L} \int_{\Gamma} e^{-ik(r+\varrho)} f(r, \varrho)\, ds\,. \tag{1.1}$$

The integrals in Eqs. (1.1) of [1] are just of this type. Let us apply the stationary phase method to the integral of Eq. (1.1), and then differentiate it. Thus, by the formula (2.2) of [1], we have:

$$J = \frac{\partial}{\partial x_L} \left(e^{-ik(\varrho+r)} f(r, \varrho) \sqrt{\frac{2\pi}{k\zeta''}}\, e^{-\frac{\pi}{4}} \right)_{\nu}. \tag{1.2}$$

The value of the expression operated on by $\partial/\partial x_L$ should be taken at point P_ν, where

$$\zeta' = \frac{d(r+\varrho)}{ds} = \frac{\vec{t}\vec{r}}{r} + \frac{\vec{t}\vec{\varrho}}{\varrho} = 0\,. \tag{1.3}$$

In carrying out the differentiation in (1.2) remember that, by Eq. (1.3), both r and ϱ (and also ξ'') depend on x_L in a different way than previously (before applying the approximate method). As $r = \sqrt{(x-x_p)^2+(y-y_p)^2+(z-z_p)^2}$, $\varrho = \sqrt{(x-x_L)^2+(y-y_L)^2+(z-z_L)^2}$, the dependence of r on x_L is involved in

the variables x, y, z, whereas ϱ depends on x_L both directly and indirectly through x, y, z.

We now carry out differentiation in (1.2), obtaining

$$J = \left(f(\varrho, r) \sqrt{\frac{2\pi}{k\zeta''}} e^{-i\frac{\pi}{4}} \right)_v \frac{\partial}{\partial x_L} e^{-ik(\varrho+r)_v} + \tag{1.4}$$

$$+ e^{-ik(r+\varrho)_v} \frac{\partial}{\partial x_L} \left(f(r, \varrho) \sqrt{\frac{2\pi}{k\zeta''}} e^{-i\frac{\pi}{4}} \right)_v .$$

We take up the term

$$\frac{\partial}{\partial x_L} e^{-ik(\varrho+r)_v} .$$

$$\frac{\partial}{\partial x_L} e^{-ik(\varrho+r)} = -ike^{-ik(\varrho+r)_v} \left[\left(\frac{\vec{t}\vec{\varrho}}{\varrho} + \frac{\vec{t}\vec{r}}{r} \right)_v \frac{\partial s}{\partial x_L} - \left(\frac{x - x_L}{\varrho} \right)_v \right],$$

whence, by Eq. (1.3),

$$\frac{\partial}{\partial x_L} e^{-ik(\varrho+r)_v} = ike^{-ik(\varrho+r)_v} \left(\frac{x - x_L}{\varrho} \right)_v . \tag{1.5}$$

Substitution of (1.5) in Eq. (1.4) yields

$$J = ike^{-ik(\varrho+r)} \frac{x - x_L}{\varrho} f(r, \varrho) \sqrt{\frac{2\pi}{k\zeta''}} e^{-i\frac{\pi}{4}} \Big|_v + \tag{1.6}$$

$$+ e^{-ik(\varrho+r)_v} \frac{\partial}{\partial x_L} \left(f(r, \varrho) \sqrt{\frac{2\pi}{k\zeta''}} e^{-i\frac{\pi}{4}} \right)_v .$$

As k is a very large quantity, a good approximation for J is obtained, if only the terms proportional to the highest power of k are retained in Eq. (1.6). Hence, finally, we have

$$J = ike^{-ik(\varrho+r)} \frac{x - x_L}{\varrho} f(r, \varrho) \sqrt{\frac{2\pi}{k\zeta''}} e^{-i\frac{\pi}{4}} \Big|_v . \tag{1.7}$$

We now turn once more to our starting point, i.e. to Eq. (1.1). In order to find the approximate value of the integral J, we shall first differentiate the expression under the integral and then proceed to apply the stationary phase method. Accordingly,

$$J = ik \int_\Gamma e^{-ik(r+\varrho)} \frac{x - x_L}{\varrho} f(r, \varrho) ds + \int_\Gamma e^{-ik(\varrho+r)} \frac{\partial f}{\partial \varrho} \frac{\partial \varrho}{\partial x_L} ds .$$

Applying formula I (2.2) to both foregoing integrals, we have

$$J = ike^{-ik(r+\varrho)} \frac{x - x_L}{\varrho} f(r, \varrho) \sqrt{\frac{2\pi}{k\zeta''}} e^{-i\frac{\pi}{4}} \Big|_v + e^{-ik(\varrho+r)} \frac{\partial f}{\partial \varrho} \frac{\partial \varrho}{\partial x_L} \sqrt{\frac{2\pi}{k\zeta''}} e^{-i\frac{\pi}{4}} \Big|_v .$$

Once more, and for the reasons stated previously, we retain the first term only, obtaining

$$J = ike^{-ik(\varrho+r)} \frac{x - x_L}{\varrho} f(r, \varrho) \sqrt{\frac{2\pi}{k\zeta''}} e^{-i\frac{\pi}{4}} \Big|_v . \tag{1.8}$$

Thus, the value of J computed by the two different procedures, as given by Eqs. (1.7) and (1.8), is the same. The result would be identical for other differential operators, as e.g. $\partial/\partial y_L$, $\partial/\partial z_L$ and for the operators $\partial/\partial x_L$, $\partial/\partial z_L$, $\partial/\partial y_L\ \partial/\partial z_L$, $\partial^2/\partial z_L^2$. In the last case, one would have to retain only the terms proportional to k^2 in the respective expressions.

Formulas (3.5) in [1] contain, moreover, expressions of the type

$$\operatorname{grad}_L \frac{\partial}{\partial z_L}\left(\frac{e^{-ikR}}{R}\int_{\infty}^{\sqrt{\frac{2k}{\pi}(r+\varrho-R)}} e^{-i\frac{\pi}{2}v^2}dv\right),\quad \operatorname{grad}_L\left(\frac{e^{-ikR}}{R}\int_{\infty}^{\sqrt{\frac{2k}{\pi}(r+\varrho-R)}} e^{-i\frac{\pi}{2}v^2}dv\right)_v \times\vec{\varkappa}.$$

Clearly, in the present case the operators of differentiation act on the upper limit of integration $\sqrt{\frac{2k}{\pi}(r+\varrho-R)}$ also. We have

$$\frac{\partial}{\partial z_L}\sqrt{\frac{2k}{\pi}(r+\varrho-R)_v} + \sqrt{\frac{k}{2\pi}}\left.\frac{\frac{z_L-z}{\varrho}+\frac{z_p-z_L}{R}}{\sqrt{r+\varrho-R}}\right|_v. \tag{1.9a}$$

To fix attention, we shall now consider the x-component of the vector $\vec{E}$ (Eq. (3.5) in [1]). First of all, by Eqs. (3.2a) in [1], the last two terms in the formula for $\vec{E}$ are found to be of order $k\sqrt{k}$. We now turn to the expression grad $\partial/\partial z_L(\ldots)$ in the formula for E. Let us ask how the operator $\partial/\partial x_L\ \partial/\partial z_L$ acts on the function

$$\left(\frac{e^{-ikR}}{R}\int_{\infty}^{\sqrt{\frac{2k}{\pi}(r+\varrho-R)}} e^{-i\frac{\pi}{2}v^2}dv\right)_v. \tag{1.10}$$

Simple considerations would lead to the conclusion that the greatest contribution to the value of the x-component of the vector $\vec{E}$ near the shadow boundary is due to the part of the "Kirchhoffian" term $\partial/\partial x_L\ \partial/\partial z_L\ (\ldots)$ proportional to k^2. The foregoing conclusions would also hold for the remaining components of the vector $\vec{E}$ and for the vector $\vec{H}$.

As can easily be proved, the expression $\partial/\partial z_L\ \sqrt{r+\varrho-R}$, has finite value at the shadow boundary, namely

$$\frac{\partial}{\partial z_L}\sqrt{r+\varrho-R} = \frac{1}{4}\sqrt{\frac{r+\varrho+R}{r\varrho}}\left[+\frac{4r^2\cos(R,\varkappa)\sin^2\left(\frac{r,\varrho}{2}\right)\cos\left(\frac{r,\varrho}{2}\right)}{R\left(R+\sqrt{R^2-r^2\sin^2(\varrho,r)}\right)}-\right.$$
$$\left.-\frac{2r}{R}\sin\left(\frac{r,\varrho}{2}\right)\sin(R,x)\cos(\varrho,R;R,x)\right]. \tag{1.9b}$$

Here, $\cos(\varrho, R; R, \varkappa)$ denotes the cosine of the angle between two planes, one of which contains the vectors $\vec{\varrho}$, $\vec{R}$ and the other — the vectors $\vec{R}, \vec{\varkappa}$ ($\vec{a}$ is the unit vector in the direction of the z-axis). This angle tends to the angle subtended by

the plane tangent to the cone of reflexion and the plane containing the vectors $\vec{R}$ and $\vec{\varkappa}$. This cone, whose axis is given by the vector $\vec{ds}$ and whose apex is the point P_v, contains all points of observation such that $\cos(r,ds) = -\cos(\vec{\varrho}, \vec{ds})$.

To prove that also the expression $\partial/\partial x_L\, \partial/\partial z_L \sqrt{r+\varrho-R}$ has a finite value within this region, we differentiate the right hand side of Eq. (1.9b) with respect to x_L.

It is easily verified that all expressions obtained are finite, with the exception perhaps of the derivative $\partial/\partial x_L \cos(\varrho, R; R, \varkappa)$. Thus, we have to prove that the expression

$$\frac{\partial}{\partial x_L}\left[\frac{\vec{\varrho}\times\vec{r}}{|\vec{\varrho}\times\vec{r}|}\cdot\frac{\vec{R}\times\vec{\varkappa}}{|\vec{R}\times\vec{\varkappa}|}\right]$$

is finite in the shadow boundary. It is sufficient to prove that the vector $\frac{\partial}{\partial x_L}\frac{\vec{\varrho}\times\vec{r}}{|\vec{\varrho}\times\vec{r}|}$ is finite, as the vector $\frac{\partial}{\partial x_L}\frac{\vec{R}\times\vec{\varkappa}}{|\vec{R}\times\vec{\varkappa}|}$ obviously is. Moreover, it should be stated that, the vector $\vec{n} = \frac{\vec{\varrho}\times\dot{r}}{|\vec{\varrho}\times\dot{r}|}$ tends to the vector perpendicular to the plane tangent to the cone of reflexion.

In order to prove that $\frac{\partial}{\partial x_L}\frac{\vec{\varrho}\times\dot{r}}{|\vec{\varrho}\times\dot{r}|}$ is finite, we introduce three mutually perpendicular axes in the direction of the vectors $\frac{\dot{r}}{r}$, $\vec{n}$, $\frac{\dot{r}}{r}\times n$.

Obviously, the following three formulas hold:

$$\vec{n}\frac{\partial\vec{n}}{\partial x_L} = 0 \quad \text{since} \quad \vec{n}^2 = 1;$$

$$\frac{\vec{r}}{r}\frac{\partial\vec{n}}{\partial x_L} = -\vec{n}\frac{\partial\frac{\dot{r}}{r}}{\partial x_L} \quad \text{since} \quad \frac{\vec{r}}{r}\vec{n} = 0;$$

$$\left(\frac{\dot{r}}{r}\times\dot{n}\right)\frac{\partial}{\partial x_L}\dot{n} = \frac{1}{|\dot{\varrho}\times\dot{r}|}\left|\left(\frac{\dot{r}\dot{\varrho}}{r}\right)\left(\dot{n}\frac{\partial\dot{r}}{\partial x_L}\right) - r\left(\dot{n}\frac{\partial\dot{\varrho}}{\partial x_L}\right)\right| \quad \text{since} \quad \dot{n}\left[\frac{\dot{r}}{r}\times\dot{n}\right] = 0.$$

The projections of the vector $\frac{\partial\vec{n}}{\partial x_L}$ on the directions n and $\frac{\vec{r}}{r}$ have finite values at the shadow boundary. We have but to prove that the projection of $\frac{\partial\vec{n}}{\partial x_L}$ in the direction $\frac{\dot{r}}{r}\times\vec{n}$ is also finite.

Taking into account that $\frac{\partial\dot{r}}{x_L} = \vec{t}\frac{\partial s}{\partial x_L}$, $\frac{\partial\dot{\varrho}}{\partial x_L} = \vec{t}\frac{\partial s}{\partial x_L} - \vec{i}$ (where $\vec{i}$ is the unit vector in the direction of the x-axis) we can write

$$\left(\frac{\vec{r}}{r}\times\dot{n}\right)\frac{\partial\vec{n}}{\partial x_L} = \frac{\vec{n}\vec{t}}{r}\frac{\partial s}{\partial x_L}\operatorname{ctg}\left(\frac{r,\varrho}{2}\right) + \frac{r(\vec{n}\vec{i}) - r(\vec{n}\vec{t})\frac{\partial s}{\partial x_L} - \varrho(\vec{n}\vec{t})\frac{\partial s}{\partial x_L}}{|\vec{\varrho}\times\vec{r}|}. \tag{1.11}$$

The term that has to be investigated, is the second one.

$$\text{With } \frac{\partial s}{\partial x_L} = -\frac{\frac{\partial F}{\partial x_L}}{\frac{\partial F}{\partial s}} = \frac{\left(\vec{t}\,\vec{i}\right) - \left(\frac{\vec{\varrho}\,\vec{t}}{\varrho}\right)\left(\frac{\vec{\varrho}\,\vec{i}}{\varrho}\right)}{\zeta''} \quad \text{where} \quad F \equiv \zeta' = \frac{\vec{r}\,\vec{t}}{r} + \frac{\vec{\varrho}\,\vec{t}}{\varrho}$$

(cf. Eq. (2.3) in [1]), it can be rewritten

$$\text{(1.12)} \quad \frac{r(\vec{n}\vec{i}) - r(\vec{n}\vec{t})\frac{\partial s}{\partial x_L} - \varrho(\vec{n}\vec{t})\frac{\partial s}{\partial x_L}}{\vec{\varrho}\times\vec{r}} = \frac{r(\vec{n}\vec{i})\left[\cos(\varrho, K) + \cos r, K)\right]}{\zeta'' K |\vec{\varrho}\times\vec{r}|} +$$

$$- \frac{\left[(-\vec{n}\vec{t})\left\{(\vec{t}\,\vec{i}) - \frac{(\vec{\varrho}\,\vec{t})}{\varrho}\frac{(\vec{\varrho}\,\vec{i})}{\varrho}\right\} + (\vec{n}\vec{i})\sin^2(\varrho, ds)\right]}{\zeta'' |\vec{\varrho}\times\vec{r}|}\frac{(r+\varrho)}{\varrho}.$$

The first right hand term in (1.12) is finite. Let us investigate the term

$$\frac{(-\vec{n}\vec{t})\left[(\vec{t}\,\vec{i}) - \frac{(\vec{\varrho}\,\vec{t})}{\varrho}\frac{(\vec{\varrho}\,\vec{i})}{\varrho}\right] + (\vec{n}\vec{i})\sin^2(\varrho, ds)}{|\vec{\varrho}\times\vec{r}|}.$$

We can easily prove that it is equal to

$$\frac{(\dot{i}\times\dot{t})\dot{\varrho}\cdot(\dot{\varrho}\,\dot{t})}{2\varrho^3 r \sin^2\left(\frac{r,\varrho}{2}\right)\zeta''}.$$

Thus, we have proved that, in the neighbourhood of the shadow boundary, the expression $\frac{\partial}{\partial x_L}\frac{\partial}{\partial z_L}\sqrt{r+\varrho-R}$ is finite. This completes the proofs required for the validity of all the conclusions concerning the orders of magnitude of the various terms in Kottler's formulas.

The foregoing considerations lead to the following conclusion of practical importance: If we are to find the electric and magnetic field strength near the shadow boundary from Eqs. (3.5) of [1], it is sufficient to take into account the terms containing the operators $\text{grad}_L \frac{\partial}{\partial z_L}$ and grad_L only, and to restrict oneself to differentiating the function $\frac{e^{-ikR}}{R}$ in the respective expressions. Thus, both the field strengths in the region considered are given by the respective derivatives of the function $\frac{e^{-ikR}}{R}$ multiplied by Fresnel's integral. Our conclusion is in full agreement with the experimental data according to which the intensity of the diffraction fringes with that computed theoretically Kirchoff's scalar theory [3].

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A Method of Solution of Potential and Temperature Equations for Conductors in a Magnetic Field and the Uniqueness Problem

by

H. STACHOWIAK

Presented by W. RUBINOWICZ on August 6, 1960

Potential and temperature equations for an electrically insulated conductor in a magnetic field for steady-state processes were given in [1]. We here write these equations in the form

$$l_1 \Delta\mu + l_{12} \Delta T + l_1' \nabla T \nabla \mu + l_{12}' (\nabla T)^2 = 0, \tag{1a}$$

$$l_{12} T \Delta\mu + l_2 T \Delta T - l_1 (\nabla\mu)^2 - [l_{12} - (l_{12} T)'] \nabla T \nabla \mu + (l_2 T)' (\nabla T)^2 = 0, \tag{1b}$$

where μ is the electrochemical potential and T — temperature. Onsager's coefficients l_1, l_2, l_{12} are functions of T only, and l_1', l_{12}' denote their derivatives.

We apply the following boundary conditions:

$$T(\sigma) = f(\sigma), \tag{2a}$$

$$\frac{\partial\mu}{\partial s_1} = -\frac{l_{12}}{l_1} \frac{\partial T}{\partial s_2}. \tag{2b}$$

Here σ is a point of the conductor surface, $\partial/\partial s_1$ and $\partial/\partial s_2$ are directional derivatives for directions defined by the formulae

$$\vec{s}_1 = \vec{n} + \frac{l_{13}}{l_1} B_v \vec{t} - \frac{l_{13}}{l_1} B_t \vec{v}, \tag{3a}$$

$$\vec{s}_2 = \vec{n} + \frac{l_{14}}{l_{12}} B_v \vec{t} - \frac{l_{14}}{l_{12}} B_t \vec{v}. \tag{3b}$$

Here $\vec{B}$ is the magnetic induction vector, $\vec{n}$ — a unit vector normal to the surface, $\vec{t}$, $\vec{v}$ — tangential and mutually orthogonal unit vectors and as regards l_{13}, l_{14} cf. [1].

If there is no magnetic field, the condition (2b) has the form

$$\frac{\partial\mu}{\partial n} = -\frac{l_{12}}{l_1} \frac{\partial T}{\partial n}. \tag{4}$$

For a given $T(\vec{r})$, $\mu(\vec{r})$ satisfying the relation

$$\nabla\mu = -\frac{l_{12}}{l_1}\nabla T \tag{5}$$

is the only solution of Eq. (1a) by the condition (4) (Neumann's problem). Introducing (5) into (1b) we obtain the equation

$$\nabla\cdot(C(T)\nabla T) = 0, \tag{6}$$

where

$$C(T) = \frac{(l_1 l_2 - l_{12}^2)T}{l_1}. \tag{7}$$

This equation has only one solution with the condition (2a). Indeed, passing from variable T to variable Θ expressed by the integral (cf. [2])

$$\Theta(\vec{r}) = \frac{1}{C(\tau)}\int_{\tau}^{T(\vec{r})} C(T)\,dT, \tag{8}$$

we obtain from (6) the equation

$$\Delta\Theta = 0, \tag{9}$$

which has one and only one solution with Dirichlet's boundary condition following from (2a).

If the magnetic field is present, μ expressed by the relation (5) does not satisfy the boundary conditions (2). We present μ in the form

$$\nabla\mu = -\frac{l_{12}}{l_1}\nabla T + \nabla\varphi. \tag{10}$$

Substituting (10) into (1) and (2) and neglecting non-linear terms in B we obtain a system of equations for φ and T. Passing from T to Θ, according to (8) we obtain the following equations

$$l_1\Delta\varphi + \dot{l}_1\nabla\Theta\nabla\varphi = 0, \tag{11a}$$

$$C(\tau)\Delta\Theta = -l_1\frac{\mathrm{d}}{\mathrm{d}\Theta}\left(\frac{\dot{l}_{12}}{l_1}\right)T(\Theta(\vec{r}))\nabla\Theta\nabla\varphi, \tag{11b}$$

where a dot over a letter denotes a derivative with respect to Θ.

We divide Θ into two parts

$$\Theta = \Theta_0 + \Theta_1, \tag{12}$$

where Θ_0 is the value of Θ, when there is no magnetic field. Neglecting non-linear terms in B and using (9) and (12) we may write

$$\Delta\Theta_0 = 0, \tag{13}$$

$$l_1\Delta\varphi + \dot{l}_1\nabla\Theta_0\Delta\varphi = 0, \tag{14}$$

$$C(\tau)\Delta\Theta_1 = -l_1\frac{\mathrm{d}}{\mathrm{d}\Theta_0}\left(\frac{\dot{l}_{12}}{l_1}\right)T(\Theta_0)\nabla\Theta_0\nabla\varphi, \tag{15}$$

with the boundary conditions

$$\Theta_0(\sigma) = \frac{1}{C(\tau)} \int_{\tau}^{f(\sigma)} C(T)\, dT, \tag{16}$$

$$\frac{\partial \varphi}{\partial n} = \frac{l_{12} l_{13} - l_1 l_{14}}{l_1^2} \left(B_v \frac{\partial f}{\partial t} - B_t \frac{\partial f}{\partial v} \right), \tag{17}$$

$$\Theta_1(\sigma) = 0. \tag{18}$$

We first compute Θ_0 from the Dirichlet problem (13), (16). Further, substituting Θ_0 into (14), we solve the problem with the normal derivative (14), (17). Finally, we substitute Θ_0 and φ into (15) and find Θ_1 from (15) and (18). Each of these three problems has only one solution (cf. [3]) disregarding a nonsubstantial additional constant in φ. The transition from Θ to T is unique, because of $C(T)$ being always positive, which follows from the second law of thermodynamics.

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One-Particle Green's Function in the Spinor Space

by
J. RZEWUSKI

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1. Introduction

The second order invariant differential operators in the spinor space may be written in the form *)

$$\frac{\partial^2}{\partial z_{\alpha;\beta}\,\partial z^{\alpha;\beta}}, \tag{1.1}$$

where

$$z_{\alpha;1} \equiv z_{\dot{\alpha};2} z_{\alpha;\dot{2}} \equiv z_{\alpha}^{\cdot}, \quad (z_{\alpha;\dot{\beta}}^{\cdot} = z_{\alpha;\beta}^{*}, \ \alpha = 1, 2) \tag{1.2}$$

are spinor variables. With respect to the index before (after) the semicolon they transform according to the unimodular group c (c'). These transformations commuting with each other, it is seen that (1.1) is invariant with respect to the direct product $c \times c'$.

Connection with Minkowski's space $z_{\alpha\dot{\beta}}$; ($z_{\alpha\dot{\beta}}$; are vectors with respect to c and scalars with respect to c') is assumed to be given by the four bilinear forms

$$z_{\alpha\dot{\beta}}; = z_{\alpha;\rho}\, z_{\dot{\beta};\dot{\sigma}}^{\cdot}\, \epsilon^{;\rho\dot{\sigma}}, \tag{1.3}$$

where $\epsilon^{;\rho\dot{\sigma}}$ is a constant vector with respect to c' and scalar with respect to c. We may choose the absolute value of its square to be unity.

It may be shown with help of (1.3) that a second order differential operator in the spinor space, corresponding with the Klein-Gordon operator in Minkowski's space, must necessarily have the form

$$D - \varkappa^2 \quad \text{or} \quad \operatorname{Re} D - \varkappa^2, \tag{1.4}$$

where

$$D = \frac{2\epsilon_{\mu}^{2}}{z_{\alpha;\dot{\beta}}^{\cdot}\, z^{\alpha;\dot{\beta}}} \frac{\partial^2}{\partial z_{\alpha;\dot{\beta}}^{\cdot}\, \partial z^{\alpha;\beta}} \tag{1.4'}$$

*) The notions and the formulae used in this section are discussed in full detail in previous papers (cf. e.g. [1] and [2]).

and Re D is the real part of D (Re $D = \frac{1}{2}(D+D^*)$). It may be noted that (1.4) is invariant with respect to a larger group of transformations than (1.1), namely with respect to the direct product $c \times c' \times a$, where a is the commutative one-parameter group defined by

$$z'_{\alpha;\beta} = e^{i\varphi} z_{\alpha;\beta}. \tag{1.5}$$

A differential equation of the type

$$(\mathrm{Re}\, D - \varkappa^2) f = \varrho, \tag{1.6}$$

where $f = f(z_{\alpha;\beta}, z_{\dot\alpha;\dot\beta})$ is the unknown function and $\varrho = \varrho(z_{\alpha;\beta}\, z_{\dot\alpha;\dot\beta})$ a given unhomogeneity, determines together with appropriate initial conditions, the function f uniquely.

A homogeneous differential equation of the type

$$(D - \varkappa^2) f = 0 \tag{1.7}$$

alone does not determine f uniquely, due to the fact that D does not contain differentiations with respect to $z_{\dot\alpha;\dot\beta} \equiv z^*_{\alpha;\beta}$. The dependence of f on $z_{\dot\alpha;\dot\beta}$ is, therefore, quite arbitrary. Also in the inhomogeneous counterpart of (1.7)

$$(D - \varkappa^2) f = \varrho, \tag{1.7'}$$

for particular ϱ's, a certain amount of arbitrariness is left, as will be seen in detail from the considerations of sec. **2**. To determine the problem completely it is, therefore, necessary to supplement Eq. (1.7) or (1.7') by an equation containing the operator D^*. Since the correspondence principle (correspondence with the Klein-Gordon equation) was already used in (1.7) and (1.7'), we have no guiding principle (apart from invariance principles) as to the form of the supplementary equation.

In this paper we shall consider in some detail equations of the form (1.7') without postulating any particular form for the supplementary equation. It will be shown that Eqs. (1.7') offer a possibility to introduce interaction in frames of linear (possibly inhomogeneous) equations in the spinor space. For this purpose it will prove convenient to write the operator D in terms of the four Minkowski variables

$$x_\mu = \sigma_\mu^{\alpha\dot\beta} z_{\alpha\dot\beta} = x_\mu(z_{\alpha;\beta}, z_{\dot\alpha;\dot\beta}) \tag{1.8}$$

and some other four variables

$$a_i = a_i(z_{\alpha;\beta}, z_{\dot\alpha;\dot\beta}) \quad (i = 1, \ldots, 4) \tag{1.9}$$

which appear to be necessarily angular variables. Eqs. (1.8)—(1.9) may be solved with respect to $z_{\alpha;\beta}$, $z_{\dot\alpha;\dot\beta}$ and we obtain easily

$$D = \frac{\partial^2}{\partial x_\mu^2} + D'\left(x_\mu,\ a_i,\ \frac{\partial}{\partial x_\mu},\ \frac{\partial}{\partial a_i}\right), \tag{1.10}$$

where D' vanishes, if acting upon a function which does not depend on a_i.

We shall base our considerations on the particularly simple case when the unknown function $f(x_\mu, \alpha_i) = f(x_\mu, \alpha)$ depends on one of the additional variables only, namely on the variable α determined by the connexion

$$e^{2i\alpha} = \frac{z_{\alpha;\beta}\, z^{\alpha;\beta}}{z_{\dot\alpha;\dot\beta}\, z^{\dot\alpha;\dot\beta}}. \tag{1.11}$$

In this case Eq. (1.7′) takes the simple form

$$\left[\left(\frac{\partial}{\partial x_\mu} - i\frac{x_\mu}{x_\sigma^2}\frac{\partial}{\partial\alpha}\right)^2 - \varkappa^2\right] f(x, \alpha) = \varrho(x, \alpha). \tag{1.12}$$

The essential features of the considerations contained in this paper remain unaltered also in the general case, when dependence on all additional variables α_i is taken into account.

The differential operator in (1.12) possesses variable coefficients. In order to preserve translational invariance with respect to the x_μ variables, we have to consider the x_μ in (1.12) as co-ordinate differences. Therefore, (1.12) may not be applied to fields which, according the their definition, are functions of co-ordinates. However, (1.12) may be applied to Green's functions or radiation operators which always are functions of co-ordinate differences only. To obtain full analogy with the conventional equation for a free-particle Green's function we finally put

$$\varrho(x, \alpha) = \delta(x). \tag{1.13}$$

2. Correspondence of the solutions of (1.12) with various types of one-particle Green's functions

Introducing (1.13) into (1.12) we get

$$\left[\left(\frac{\partial}{\partial x_\mu} - i\frac{x_\mu}{x_\sigma^2}\frac{\partial}{\partial\alpha}\right)^2 - \varkappa^2\right] f(x, \alpha) = \delta(x), \tag{2.1}$$

where, in general, $\varkappa^2$ may be considered as a function of x_μ and α. If $\varkappa^2$ depends on x_μ and α only by the intermediary of one complex variable

$$z = \frac{e^{i\alpha}}{\lambda\sqrt{|x_\mu^2|}} \tag{2.2}$$

then the general solution of (2.1) may be written in the form

$$f(x, \alpha) = F(x, z). \tag{2.3}$$

Here $F(x, z)$ is an arbitrary function of z which satisfies equation

$$(\Box^{\text{expl}} - \varkappa^2)\, F(x, z) = \delta(x) \tag{2.4}$$

with respect to the variables x_μ occurring explicitly in $F(x, z)$ (not by the intermediary of z). The arbitrariness in the functional dependence on z is connected with the arbitrary dependence on $z_{\dot\alpha;\dot\beta}$ of the solutions of the full equation (1.7′)

mentioned in sec. **1**. It may be removed by a supplementary equation in z. The parameter λ with dimension of length is introduced for dimensional reasons. Its rôle will become clear presently.

To interpret solution (2.3) physically we average over the variable α

$$\bar{F}(x;\lambda) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x,z)\,d\alpha \tag{2.5}$$

so that to every solution $F(x,z)$ there corresponds a function $\bar{F}(x;\lambda)$ of the space-time variables only. We are going to show that by appropriate choice of the arbitrary functional dependence of $F(x,z)$ on z we may obtain for $\bar{F}(x;\lambda)$ the various types of one-particle Green's functions as, e.g., Green's function of a non-interacting particle, the regularized Green's function or Green's function of a particle interacting virtually with other fields.

We note that integration (2.5) may be conveniently written in form of a contour integration in the complex z-plane,

$$\bar{F}(x;\lambda) = \frac{1}{2\pi i}\oint F(x,z)\frac{dz}{z}, \tag{2.6}$$

where the contour of integration is a circle around the point $z=0$ $(x_\mu^2 = \infty)$ with the radius

$$|z| = (\lambda\sqrt{|x_\mu^2|})^{-1}. \tag{2.7}$$

We may also write solution (2.3) in the form

$$F(x,z) = \Delta(x,\varkappa^2) f(z), \tag{2.8}$$

where $\Delta(x,\varkappa^2)$ satisfies Eq. (2.4) with $\varkappa^2(z)$ an arbitrary function of z and $f(z)$ another arbitrary function of z satisfying condition

$$f(\infty) = 1. \tag{2.9}$$

Indeed, introducing (2.8) into (2.1) we obtain

$$\left[\left(\frac{\partial}{\partial x_\mu} - i\frac{x_\mu}{x_\sigma^2}\frac{\partial}{\partial \varkappa^2}\right)^2 - \varkappa^2\right]\Delta(x,\alpha^2) f(z) = \delta(x) f(z) = \delta(x) f(\infty) = \delta(x). \tag{2.10}$$

Consider now the simplest case when $f(z) = 1$. In this case the contour integration in (2.6) yields the residue of the function $\Delta(x,\varkappa^2) z^{-1}$ at the point $z=0$, i.e. $\Delta(x,\varkappa_0^2)$ which is the ordinary Green's function of a free boson field with squared mass equal to $\varkappa_0^2 = \varkappa^2(0)$.

The next simple example arises when $f(z)$ has some poles

$$f(z) = \frac{\varphi(z)}{\prod_i\left(1 - \frac{z}{z_i}\right)}, \tag{2.11}$$

where $\varphi(z)$ is a polynomial of the same order as the denominator, so that $f(z) \to 1$ as $z \to \infty$. Introducing (2.8) with (2.11) into (2.6) we obtain:

$$\bar{F}(x;\lambda) = \frac{1}{2\pi i}\oint \frac{\Delta(x,\varkappa^2)\,\varphi(z)}{z\prod\limits_i\left(1-\frac{z}{z_i}\right)}\,dz = \varphi(0)\,\Delta(x,\varkappa_0^2) - \sum_{k=1}^{n(\lambda\sqrt{|x_\mu^2|})} \frac{\varphi(z_k)\,\Delta(x;\varkappa_k^2)}{\prod\limits_i{}'\left(1-\frac{z_k}{z_i}\right)}, \tag{2.12}$$

where $\varkappa_k^2 = \varkappa^2(z_k)$. The number of terms $n(\lambda\sqrt{|x_\mu^2|})$ is a step-function of $\lambda\sqrt{|x_\mu^2|}$ which increases as x_μ approaches the light cone $x_\mu^2 = 0$. (2.12) represents obviously a modification of the Pauli-Villars method of regularization [3]. For sufficiently large $|x_\mu^2|$ we have the unregularized function $\varphi(0)\,\Delta(x,\varkappa_0^2)$. As we approach the light cone, more and more terms in the sum appear. Sufficiently near to the light cone all poles are included and we have full regularization if the arbitrary coefficients

$$\varphi(0),\quad \frac{\varphi(z_k)}{\prod\limits_i{}'\left(1-\frac{z_k}{z_i}\right)},\quad (k=1,2,\ldots N = \text{number of poles}) \tag{2.13}$$

are chosen in such a way as to satisfy the well known regularity conditions of Pauli-Villars. It may be easily shown that the first condition

$$\varphi(0) + \sum_{k=1}^{N} \frac{\varphi(z_k)}{\prod\limits_{i=1}^{N}{}'\left(1-\frac{z_k}{z_i}\right)} = 0 \tag{2.14}$$

is automatically satisfied if $\varphi(0) = \varphi(z_1) = \varphi(z_2) = \ldots = \varphi(z_N) = 1$. We note further that in the limit $\lambda \to 0$ all poles are included for every finite x_μ^2 and we have full regularization everywhere.

Let us consider, finally, the case when $f(z)$ in (2.8) has a cut along the real axis beginning at $r \equiv |z| = r_0$ and extending to ∞. The integral (2.6) becomes now

$$\bar{F}(x;\lambda) = \frac{1}{2\pi i}\oint \Delta(x,\varkappa^2) f(z)\frac{dz}{z} = f(0)\,\Delta(x,\varkappa_0^2) - \frac{1}{2\pi i}\int_{r_0}^{(\lambda\sqrt{|x_\mu^2|})-1} \Delta(x,\varkappa_r^2)\,\frac{f(r+i\epsilon)-f(r-i\epsilon)}{r}\,dr, \tag{2.15}$$

where $\varkappa_0^2 = \varkappa^2(0)$, $\varkappa_r^2 = \varkappa^2(r)$ and $\varepsilon > 0$, $\varepsilon \to 0$. Assuming the functional dependence of $\varkappa^2$ on r to be linear in the region (r_0, ∞)

$$\varkappa^2(r) = r, \tag{2.16}$$

and putting $f(0) = 1$, we finally obtain

$$\overline{F}(x;\lambda) = \Delta(x, \varkappa_0^2) + \int_{r_0}^{(\lambda\sqrt{|x_\mu^2|})^{-1}} \Delta(x, r)\, \varrho(r)\, dr \tag{2.17}$$

with

$$\varrho(r) = \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \frac{1}{2\pi i} \frac{f(r - i\epsilon) - f(r + i\epsilon)}{r}. \tag{2.18}$$

This function possesses the form of a spectral representation for the one-particle Green's function (cf. [4] and [5]) with an upper limit in the integral, cutting off the contributions corresponding to large r for large distances x_μ^2. The conventional form is obtained for $\lambda \to 0$.

Summary

The solutions of the second order invariant differential equation in the spinor space are investigated with respect to their correspondence to the solutions of one-particle problems in quantum field theory. It is found that the manifold of solutions of linear equations in the spinor space contains functions which correspond to solutions of non-linear equations in Minkowski's space. This seems to offer a new approach to the problem of interaction.

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Two-Particle Green's Function in the Spinor Space

by
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Equations for the two-particle problem

The radiation operator or the Green's function of a two-particle problem is a function of three co-ordinate differences and should satisfy equations in which products of two differential operators of the type (2.1) (cf. preceding paper [1] referred to as I throughout this paper) occur. We therefore postulate the symmetric set of equations*)

$$D_{x_1\alpha_1} D_{x_2\alpha_2} f_3(x_1, \alpha_1;\ x_2, \alpha_2) = \delta(x_1)\,\delta(x_2)$$

(1.1) $$D_{x_2\alpha_2} D_{x_3\alpha_3} f_1(x_2, \alpha_2;\ x_3, \alpha_3) = \delta(x_2)\,\delta(x_3)$$

$$D_{x_3\alpha_3} D_{x_1\alpha_1} f_2(x_3, \alpha_3;\ x_1, \alpha_1) = \delta(x_3)\,\delta(x_1),$$

where

(1.2) $$D_{x\alpha} \equiv \left(\frac{\partial}{\partial x_\mu} - i\frac{x_\mu}{x_\sigma^2}\,\frac{\partial}{\partial \alpha}\right)^2 - \varkappa^2.$$

The solutions of (1.1) may be written in the form

(1.3) $$F_i(x_j, z_j;\ x_k, z_k).$$

where

(1.4) $$z_k = \frac{e^{i\alpha_k}}{\lambda\sqrt{|x_{k\mu}^2|}},$$

if the function F_i satisfies the equation

(1.5) $$(\Box_j^{\mathrm{expl}} - \varkappa_j^2)(\Box_k^{\mathrm{expl}} - \varkappa_k^2)\,F_i(x_j, z_j;\ x_k, z_k) = \delta(x_j)\,\delta(x_k),$$

with respect to the $x_{j\mu}$ and $x_{k\mu}$ occurring explicitly in F_i (not by means of z_j and z_k). The dependence of $F_i(x_j, z_j;\ x_k, z_k)$ on z_j and z_k is arbitrary. We may further write (1.3) in the particular form

(1.6) $$F_i(x_j, z_j;\ x_k, z_k) = \Delta(x_j, \varkappa_j^2)\,\Delta(x_k, \varkappa_k^2)\,f_i(z_j, z_k),$$

*) The notions and the formulae used in this section are discussed in full detail in previous works (cf.e.g. [1] and [2]).

where $\Delta(x_j, \varkappa_j^2)$ is a solution of (I.2.4) with $\varkappa_j^2(z_j)$ an arbitrary function of z_j, and $f_j(z_j, z_k)$ an arbitrary function of two complex variables z_j and z_k, satisfying the boundary condition

$$f_i(\infty, \infty) = 1. \tag{1.7}$$

To get physical interpretation of the solution (1.6) we average over the variables a_j and a_k

$$\bar{F}_i(x_j, x_k; \lambda) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} da_j \int_{-\pi}^{\pi} da_k \Delta(x_j, \varkappa_j^2) \Delta(x_k, \varkappa_k^2) f_i(z_j, z_k) = \tag{1.8}$$

$$= \frac{1}{(2\pi i)^2} \oint \oint \Delta(x_j, \varkappa_j^2) \Delta(x_k, \varkappa_k^2) f_i(z_j, z_k) \frac{dz_j\, dz_k}{z_j\, z_k}.$$

Here, the contour integrations are along circles in the complex z_j and z_k planes with centers at $z_j = 0$ and $z_k = 0$, respectively, and with radii

$$(\lambda \sqrt{|x_{j\mu}^2|})^{-1} \quad \text{and} \quad (\lambda \sqrt{|x_{k\mu}^2|})^{-1} \text{ resp.} \tag{1.9}$$

In analogy with the one-particle case discussed in sec. 2 of I we obtain here the various Green's functions or radiation operators of the two-particle problem. In the case when $f_i(z_j, z_k)$ does not depend on z_j and z_k we have $f_i(z_j, z_k) = 1$ and $\bar{F}_i(x_j, x_k)$ is simply the product of two Green's functions of free particles corresponding to the squared masses $\varkappa_j^2(0)$ and $\varkappa_k^2(0)$. In the case when $f(z_j, z_k)$ possesses some poles in the complex z_j and z_k planes, the result $F_i(x_j, x_k; \lambda)$ of the integration (1.8) is a product of two regularized free-particle Green's functions with the modified regularization prescription.

Let us now consider in some detail the case when $f(z_j, z_k)$ possesses cuts in the z_j and z_k planes situated on the real positive axes and ranging from r_{j0} to ∞ and r_{k0} to ∞ resp. ($r = |z|$). Integration (1.8) yields in this case

$$\bar{F}_i(x_j, x_k; \lambda) = f_i(0,0) \Delta(x_j, \varkappa_{j0}^2) \Delta(x_k, \varkappa_{k0}^2) - \tag{1.10}$$

$$- \frac{1}{2\pi i} \int_{r_{j0}}^{(\lambda \sqrt{|x_{j\mu}^2|})^{-1}} \Delta(x_j, \varkappa_j^2) \Delta(x_k, \varkappa_{k0}^2) \frac{f_i(r_j^+, 0) - f_i(r_j^-, 0)}{r_j} dr_j -$$

$$- \frac{1}{2\pi i} \int_{r_{k0}}^{(\lambda \sqrt{|x_{k\mu}^2|})^{-1}} \Delta(x_j, \varkappa_{j0}^2) \Delta(x_k, \varkappa_k^2) \frac{f_i(0, r_k^+) - f_i(0, r_k^-)}{r_k} dr_k + \frac{1}{(2\pi i)^2} \int_{r_{j0}}^{(\lambda \sqrt{|x_{j\mu}^2|})^{-1}} \Delta(x_j, \varkappa_j^2) \times$$

$$\times \int_{r_{k0}}^{(\lambda \sqrt{|x_{k\mu}^2|})^{-1}} \Delta(x_k, \varkappa_k^2) \frac{f_i(r_j^+, r_k^+) - f_i(r_j^+, r_k^-) - f_i(r_j^-, r_k^+) + f_i(r_j^-, r_k^-)}{r_j\, r_k} dr_j\, dr_k,$$

where $r_j^{\pm} = r_j \pm i\epsilon$ ($\epsilon > 0$, $\epsilon \to 0$). In general, the position r_{j0} at which the cut begins depends upon the value of the other variable r_k and *vice versa*. Therefore, r_{j0} and r_{k0} in the integrals (1.10) are to be understood as lower limits, the assumption being made that such lower limits exist. Denoting

$$(1.11)\qquad f_i(0,0) = f_{i0}, \qquad \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} \frac{1}{2\pi i}\,\frac{f_i(r_j^-,0) - f_i(r_j^+,0)}{r_j} = f_{i1}(r_j),$$

$$\lim_{\substack{\epsilon\to 0\\ \epsilon>0}} \frac{1}{2\pi i}\,\frac{f_i(0,r_k^-) - f_i(0,r_k^+)}{r_k} = f_{i2}(r_k),$$

$$\lim_{\substack{\epsilon\to 0\\ \epsilon>0}} \frac{1}{(2\pi i)^2}\,\frac{f_i(r_j^-,r_k^-) - f_i(r_j^-,r_k^+) - f_i(r_j^+,r_k^-) + f_i(r_j^+,r_k^+)}{r_j r_k} = f_{jk}(r_j, r_k),$$

and assuming in the intervals (r_{i_0}, ∞) linear connections $\varkappa_j^2(r_i) = r_i$, we may write Eqs. (1.10) in the simple form

$$(1.12)\qquad \bar{F}_i(x_j, x_k; \lambda) = f_{i0}\,\Delta(x_j, \varkappa_j^2)\,\Delta(x_k, \varkappa_{k0}^2) +$$

$$+\Delta(x_k, \varkappa_{k0}^2) \int_{r_{j0}}^{(\lambda\,|\sqrt{x_{j\mu}^2}|)^{-1}} \Delta(x_j, r_j) f_{i1}(r_j)\,dr_j + \Delta(x_j, \varkappa_{j0}^2) \int_{r_{k0}}^{(\lambda\,|\sqrt{x_{k\mu}^2}|)^{-1}} \Delta(x_k, r_k) f_{i2}(r_k)\,dr_k +$$

$$+ \int_{r_{j0}}^{(\lambda\,|\sqrt{x_{j\mu}^2}|)^{-1}} \Delta(x_j, r_k) \int_{r_{k0}}^{(\lambda\,|\sqrt{x_{k\mu}^2}|)^{-1}} \Delta(x_k, r_k) f_{jk}(r_j, r_k)\,dr_j\,dr_k .$$

These averaged solutions of the set (1.1) may be directly connected with Green's functions as well as with the radiation operators of the problem of two interacting particles. We shall consider here only the second case.

The statement may be formulated as follows: for appropriate choice of f_{i0}, f_{i1}, f_{i2} and f_{jk} and for $\lambda \to 0$, the function

$$(1.13)\qquad R(x_1, x_2, x_3; \lambda) = \delta(x_1) F_1(x_2, x_3; \lambda) + \delta(x_2) F_2(x_3, x_1; \lambda) + \delta(x_3) F_3(x_1, x_2; \lambda),$$

represents that part of the radiation operator

$$(1.14)\qquad R'(\xi_1, \xi_2, \xi_3, \xi_4) =$$

$$= \left\langle 0 \left| \frac{\delta^4 S}{\delta\varphi_1(\xi_1)\,\delta\varphi_2(\xi_2)\,\delta\varphi_3(\xi_3)\,\delta\varphi_4(\xi_4)}\, S^+ \right| 0 \right\rangle \qquad (S = S\text{—matrix}),$$

which determines the process with two incoming and two outgoing real particles.

Indeed, putting $f_{i0} = f_{i1} = f_{i2} = 0$, $\lambda = 0$ and transforming (1.13) to momentum space we get, on account of (1.12).

$$(1.15)\qquad R(q_1, q_2, q_3) = \int\int\int dx_1\, dx_2\, dx_3\, R(x_1, x_2, x_3; 0)\, e^{-i(q_1 x_1 + q_2 x_2 + q_3 x_3)} =$$

$$= \int\int dx_2\, dx_3\, F_1(x_2, x_3; 0)\, e^{-i(q_2 x_2 + q_3 x_3)} + \int\int dx_3\, dx_1\, F_2(x_3, x_1; 0)\, e^{-i(q_3 x_3 + q_1 x_1)} +$$

$$+ \int\int dx_1\, dx_2\, F_3(x_1, x_2; 0)\, e^{-i(q_1 x_1 + q_2 x_2)} = \int_{r_{20}}^{\infty} dr_2 \int_{r_{30}}^{\infty} dr_3 \frac{f_{23}(r_2, r_3)}{(r_2 + q_{2\mu}^2)(r_3 + q_{3\mu}^2)} +$$

$$+ \int_{r_{30}}^{\infty} dr_3 \int_{r_{10}}^{\infty} dr_1 \frac{f_{31}(r_3, r_1)}{(r_3 + q_{3\mu}^2)(r_1 + q_{1\mu}^2)} + \int_{r_{10}}^{\infty} dr_1 \int_{r_{20}}^{\infty} dr_2 \frac{f_{12}(r_1, r_2)}{(r_1 + q_{1\mu}^2)(r_2 + q_{2\mu}^2)}.$$

Here, the formula

$$(1.16)\qquad \Delta(x, r) = -\frac{1}{(2\pi)^4} \int \frac{e^{iqx}}{r + q_\mu^2}\, dx$$

has been used. Taking for the solution of (I.2.4) the causal function $\Delta^c(x, r)$ we obtain in (1.15) an expression which possesses the form of Mandelstam's representation [2] for the corresponding scattering amplitude if the q_i satisfy the relation

$$(1.17)\qquad q_{1\mu}^2 + q_{2\mu}^2 + q_{3\mu}^2 = -\sum_{i=1}^{4} M_i^2,$$

where M_i $(i = 1, \ldots, 4)$ are the masses of the four particles taking part in the reaction.

On the other hand, the scattering amplitude (1.15) is connected with the Fourier transform of (1.14) in the following way

$$(1.18)\qquad R'(p_1, p_2, p_3, p_4) =$$

$$= \int\int\int\int d\xi_1\, d\xi_2\, d\xi_3\, d\xi_4\, R'(\xi_1, \xi_2, \xi_3, \xi_4)\, e^{-i(p_1\xi_1 + p_2\xi_2 - p_3\xi_3 + p_4\xi_4)} =$$

$$= 2(2\pi)^4\, \delta(q_4) \int\int\int dx_1\, dx_2\, dx_3\, R'(x_1, x_2, x_3)\, e^{-i(q_1 x_1 + q_2 x_2 + q_3 x_3)} =$$

$$= 2(2\pi)\, \delta(q_4)\, R'(q_1, q_2, q_3),$$

where

$$(1.19)\qquad \begin{cases} q_1 = p_1 + p_2, \\ q_2 = p_1 \quad + p_3 \\ q_3 = p_1 \quad\quad + p_4. \\ p_4 = p_1 + p_2 + p_3 + p_4, \end{cases} \qquad (1.20)\qquad \begin{cases} 2x_1 = \xi_1 + \xi_2 - \xi_3 - \xi_4, \\ 2x_2 = \xi_1 - \xi_2 + \xi_3 - \xi_4, \\ 2x_3 = \xi_1 - \xi_2 - \xi_3 + \xi_4, \\ 2x_4 = -\xi_1 + \xi_2 + \xi_3 + \xi_4. \end{cases}$$

The fact that R' finally appears as function of three co-ordinate differences is, of course, due to translational invariance.

For such real scattering processes for which the Mandelstam representation is valid, one may always choose $f_{ik}(r_i, r_k)$ in (1.15) in such a way that

$$(1.21)\qquad R'(q_1, q_2, q_3) = R(q_1, q_2, q_3).$$

Comparing (1.15) with (1.18) it is seen that in this case the function $R(x_1, x_2, x_3; 0)$ yields the same contribution to the scattering amplitude as the radiation operator $R'(\xi_1, \xi_2, \xi_3, \xi_4) = R'(x_1, x_2, x_3)$. They may differ by a function whose Fourier transform of the type (1.15), with q_1, q_2, q_3 satisfying relation (1.17), vanishes.

Conclusion

The content of the preceding section and of sec. 2 of I shows that the manifold of solutions of the linear equations (I.2.1) and (1.1) contains functions which, upon averaging over the angular variables α_i, and in the limit $\lambda \to 0$, go over into the various solutions of the one- and two-particle problem in conventional quantum field theory. It remains an open question as to whether the same holds also for more complicated problems. To choose the right solutions out of all solutions of (I.2.1) or (1.1) one has to determine the arbitrary dependence of these solutions on the complex variable z, which may be done e.g. by postulating appropriate equations in z for $F(x, z)$ and $F_i(x_j, z_j; x_k, z_k)$. This seems to be a very difficult problem in view of the difficulties already encountered in determining the spectral functio ns in quantum field theory. However it is, the considerations of this paper seem to indicate another possible approach to the problem of interaction. Instead of the set of non-linear integro-differential equations in Minkowski's space, which have resisted so far all attempts to find their exact solutions, one may use for Green's functions a simple set of linear equations in the spinor space which possesses immediate solutions, and support this set by certain supplementary equations determining the z-dependence.

Another point is that we have always the possibility to introduce regularized functions by introducing additional poles in the z plane. This way of introducing formactors into the theory does not contradict the principle of microscopic causality.

Finally, it may be noted that before transition to the limiting case $\lambda \to 0$ all sums are finite and all integrals possess an upper limit $(\lambda \sqrt{|x_\mu^2|})^{-1}$ which is finite, unless for points on the light cone. It is seen from (I.2.17) and (1.12) that this upper limit cuts off the interaction for large x_μ^2. In other words, the connection (I.1.3) of the spinor space and Minkowski's space introduces automatically a certain kind of adiabatic switching off of interaction. The distance for which all interaction is switched off is determined by the value of the parameter λ and the lower limit in the corresponding integral: $|x_\mu^2| = (\lambda r_0)^{-2}$.

Summary

The solutions of a system of second order invariant differential equations in the spinor space are investigated with respect to their correspondence to the solutions of two-particle problems in quantum field theory. It is shown that the manifold of solutions of linear equations in the spinor space contains functions which correspond to the two-particle Green's functions in Minkowski's space.

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THEORETICAL PHYSICS

Some Questions of Relativistic Cosmology

by
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A new form of relativistic cosmology based on three plausible principles was built up by the author in [1]. The aim of this paper is to state clearly its assumptions. A simple formula will be deduced with the help of which empirical data can unambiguously decide the question whether the universe is oscillating or expanding to infinity. The last three paragraphs deal with the forms of matter in the universe.

1. Dynamical law of the universe. This law was deduced [1] in full agreement with the ideas of the general theory of relativity under the assumption that the curvature of space on the boundary of each of the N identical Schwarzschild's cells into which the universe is considered to be divided, coincides here with the mean curvature of the universe. This law is expressed by the equation

$$\left(\frac{1}{cR}\frac{dR}{dt}\right)^2 + \frac{1}{R^2} = \frac{8\pi PK}{3c^2}\varrho, \tag{1.1}$$

where c is the velocity of light, K — Newton's gravitation constant, and P — an unimportant dimensionless constant for which we may put 1.

The first term on the left-hand side of Eq. (1.1) is the square of the Hubble's factor of cosmical expansion H divided by c. The mean density of all the forms of matter in the universe which cause the curvature of space is denoted by ϱ. If

$$H^2 < (8\pi PK/3)\,\varrho, \tag{1.2a}$$

and if $R(t)$ denotes the radius of the universe at the moment in which H and ϱ were measured, then $R(t)$ computed by means of Eq. (1.1) is real and the universe is therefore spherical. If

$$H^2 > (8\pi PK/3)\,\varrho, \tag{1.2b}$$

this radius is imaginary and the universe pseudospherical. However, since the estimation of the mean density ϱ of matter in the universe is uncertain, the relations (1.2a, b) cannot be used for determining the form of expansion of the universe.

2. Density of matter in the universe. For integrating Eq. (1.1) we must know the dependence of the density ϱ on the radius of the universe. The most general form of this function is

$$\varrho = \varrho_1 R_1^3/R^3 + \Pi, \tag{2.1}$$

where

$$\varrho_1 = \varrho(t_1), \quad R_1 = R(t_1). \tag{2.2}$$

If $\Pi = 0$, the total mass in the universe remains constant.

Contrary to the opinion expressed in [1], the function Π must not be excluded entirely from further considerations. We exclude only the function Π depending explicitely on time, because this would imply a spontaneous act of creation of matter from nothing which stands out of all natural laws. However, we may assume $\Pi = \Pi(R)$, for this is equivalent to the assumption that in the universe such a form of matter exists which, under certain conditions (for instance, in the regions of extremely high density), does not influence the curvature of space. In this case the function $\Pi(R)$ does not describe the creation of matter from nothing, but merely its transformation from a yet unknown form, not causing the curvature of space, into one doing it. As will be shown below, present-day science gives no sufficient reasons justifying the introduction of this Π-process of transformation of matter into relativistic cosmology.

3. The form of the expansion of the universe. In order to decide this question we deduce from Eqs. (4.15) and (4.16) of [1] a formula determining the time T that elapsed from the beginning of the expansion of a spherical universe to the present day. We find (assuming $\Pi = 0$)

$$T = \Omega(x)/H, \tag{3.1}$$

where

$$x = H^2/[(8\pi PK/3)\varrho - H^2], \tag{3.2}$$

$$\Omega(x) = (1+x)\sqrt{x}\ \operatorname{arc\,ctg}\sqrt{x} - x. \tag{3.3}$$

In the beginning of the expansion of a spherical universe

$$x = \infty, \quad \lim_{x\to\infty} \Omega(x) = \frac{2}{3}. \tag{3.4a}$$

During the subsequent expansion the function $\Omega(x)$ decreases monotonously to zero which is reached at the moment of the maximal expansion when

$$x = 0, \quad \lim_{x\to 0} \Omega(x) = 0. \tag{3.4b}$$

The asymptotic expansion of the function $\Omega(x)$ valid for $|x| > 1$ is

$$\Omega(x) = 2 \sum_{k=0}^{\infty} \frac{(-x)^{-k}}{(2k+1)(2k+3)}. \tag{3.5}$$

Eqs. (3.1), (3.2), (3.3) and (3.5) hold also for a pseudospherical universe when x takes negative value *). At the beginning of expansion

$$x = -\infty, \quad \lim_{x \to -\infty} \Omega(x) = \frac{2}{3} \tag{3.6a}$$

and at its limit, reached after infinite time will have elapsed

$$x = -1, \quad \lim_{x = -1} \Omega(x) = 1. \tag{3.6b}$$

Hence, we have obtained a very simple and available criterion : the universe is spherical (i.e. oscillating), if

$$TH < \frac{2}{3}, \tag{3.7a}$$

and it is pseudospherical (i.e. expanding to infinity), if

$$TH > \frac{2}{3}. \tag{3.7b}$$

For completeness we transcribe here also Eqs. (4.15) and (4.11) of [1]. The half-period T_0 of an oscillating universe, i.e. the time from the beginning of expansion to its maximum, is given by

$$T_0 = (\pi/2)(1+x)\sqrt{x}/H. \tag{3.8}$$

The maximal radius of the oscillating universe R_0 is determined by

$$R_0 = (2c/\pi)\, T_0 = c\,(1+x)\sqrt{x}/H. \tag{3.9}$$

4. The steady-state universe was proposed by Bondi and Gold [2], by Hoyle [3], and by McCrea [4] in order to explain why the observed properties of the universe do not depend on the distance from our galaxy, i.e. on the observed epoch of the universe. The aim of this note is to examine whether the numerical data of the mean density of matter obtained by observations unavoidably testify in favour of the steady-state model.

Let us suppose the mean density in the universe to be 10^{-28} g/cm^3, and the Hubble's factor $H = 5.3 \times 10^{-18}$ sec.$^{-1}$ (i.e. $1/H = 6 \times 10^9$ years). We find then: $x \cong 1$, the age of the universe $T \cong 3.5 \times 10^9$ years, and its present-day radius $R \cong 6 \times 10^9$ light years.

We now assume the universe to be accessible for astronomical observations up to the distance of 10^9 light years. The radius R_1 of the universe 10^9 years ago computed with the help of Eqs. (4.5b) of the paper [1] was 3×10^9 light-years. With regard to Eq. (2.1) (wherein we now put $\Pi = 0$), the observed mean density in the regions 10^9 light-years distant should thus amount to 8-multiple of the density in the neighbourhood of our galaxy.*).

*) The proof will be given in a subsequent paper of the author.

For the mean density we have taken its upper limit. In fact, it has very probably a lower value due to which also the ratio of both densities under consideration is smaller than 8 (especially if the universe is pseudospherical).

Considering the uncertainty in estimation of the mean density of matter in the universe ($\varrho \cong 10^{-29} \div 10^{-28}$ g/cm^3) we should conclude that there is no sufficient reason for introducing the highly hypothetical idea of a continuous creation of matter from nothing (or, in our interpretation, of a continuous transformation of matter described by the function $\Pi(R)$) into the relativistic cosmology.

5. On the hypothesis of the real existence of Hubble's mass. As follows from Eq. (1.1) a part of the total density ϱ is counterbalanced by the first term on the left-hand side, and the other by the second term. In [1] the former was called Hubble's density ϱ_H

$$\varrho_H = (3/8\pi PK)\, H^2, \tag{5.1}$$

and the latter denoted by ϱ_R:

$$\varrho_R = (3\, c^2/8\pi PK)/R^2. \tag{5.2}$$

Assuming $\Pi = 0$, the total mass M in the universe remains constant during the expansion, while the masses corresponding to ϱ_H and ϱ_R change:

$$M_H = [1 - (R/R_0)]\, M, \tag{5.3}$$

$$M_R = (R/R_0)\, M. \tag{5.4}$$

There arises now the question whether the density of the rest mass and of other forms of matter is to be identified with ϱ_R (as it is done in the hypothesis in [1]) or with ϱ. In the former case the universe is certainly spherical and Hubble's mass is a real physically entity. In the latter case the universe may be either spherical ($\varrho > \varrho_H$) or pseudospherical ($\varrho < \varrho_H$), and Hubble's mass is a mere mathematical substitution as well as the density ϱ_R. The relations deduced in the present paper and in [1] hold, however, in both cases.

There are two reasons in favour of the hypothesis:

(1) It is in full agreement with the ideas of the theory of relativity to join an increase to the rest mass of every moving body. Our hypothesis supposes that the receding of nebulae is also connected with an increase of their masses.

(2) According to Eq. (5.4) the total rest mass M in the universe vanishes at the beginning of expansion even without introducing the highly hypothetical Π-process of transformation of matter.

6. Some consequences from the theory of geodesics. The theory of geodesics in the expanding universe was developed by Tolman [5] and recently by Schrödinger [6]. We take over their results with the only difference that the integration constant has been determined by the initial condition.

A test body with the initial velocity u_1 measured in an ordinary manner by a local observer remaining at rest with respect to the average motion of matter

in his neighbourhood, changes its velocity during the expansion of the universe accordingly to the equation

$$u(R) = \frac{u_1 \frac{R_1}{R}}{\sqrt{1 - \frac{u_1^2}{c^2}\left(1 - \frac{R_1^2}{R^2}\right)}}. \tag{6.1}$$

Similarly, denoting by m_R (R_1) the rest mass of a body at the radius R_1 of the universe we find that it changes owing to the motion of the body and to the change of the radius of the universe conformly to the formula

$$m_{R+K}(R) = m_R(R_1)\sqrt{1 + \frac{u_1^2}{c^2 - u_1^2}\frac{R_1^2}{R^2}}. \tag{6.2}$$

If we consider a photon, its velocity, computed by Eq. (6.1) wherein we put $u_1 = c$, does not change

$$u_{Ph} = c, \tag{6.3}$$

but its mass, and consequently, also its frequency change as follows from Eq. (6.2) by the limiting process $u_1 \to c$, conformly to the relation

$$m_{Ph}(R)/m_{Ph}(R_1) = \nu(R)/\nu(R_1) = R_1/R. \tag{6.4}$$

The relations (6.1), (6.2), and (6.4) show that the velocity and the mass of a test body and the inertial mass of a photon decrease during the expansion of the universe and increase during its contraction. In the very late state of contraction, when $R \to 0$, the masses m_{R+K} and m_{Ph} tend to infinity.

These divergencies can be removed, if we apply here our hypothesis according to which the masses m_{R+K} (R) and m_{Ph} (R) should depend on the radius of the universe by Eq. (5.4). We have then:

$$m_{R+K}(R) = \sqrt{\left[m_R(R_1)\frac{R}{R_0}\right]^2 + \left[m_R(R_1)\frac{R_1}{R_0}\right]^2 \frac{u_1^2}{c^2 - u_1^2}}, \tag{6.5}$$

$$m_{Ph}(R) = m_{Ph}(R_1)\,[R_1/R_0]. \tag{6.6}$$

The first term on the right-hand side of Eq. (6.5) is the square of the rest mass which changes during the expansion and contraction of the universe. The second term represents the increase of mass of the body caused by its motion. This term remains constant.

The right-hand side of Eq. (6.6) is the inertial mass of a photon reduced on the maximal radius of the universe by Eq. (6.4). It also does not change. Eq. (6.6) testifies that the red shift of spectral lines emitted by nebulae is really a Doppler effect and not — as Schrödinger [6] believes on the basis of the equation which we have denoted (6.4) — something that happens to the light on its journey, along with the expansion of space.

Thus, thanks to our hypothesis, the well-known difficulties of the mysterious piston which absorbs energy in the expanding universe are overcome. This fact may be considered as the third reason in favour of our hypothesis.

Conclusions

(1) The dynamics of the universe developed in [1] and supplemented by the first three paragraphs of this paper, forms a solid basis for cosmological investigations and is reliable in the same degree as the general relativity theory. The relations (3.7a, b) deduced without any additional hypothesis give a very simple criterion for answering the question whether the universe is oscillating or expanding to infinity.

(2) There is no sufficient reason for introducing the highly hypothetical idea of a continuous creation of matter from nothing (or, in our interpretation, of a Π-process of transformation of matter) into the relativistic cosmology.

(3) Although the hypothesis of the real existence of Hubble's mass has been effective in solving some problems of relativistic cosmology, it is, however, only a hypothesis which needs empirical tests and deeper theoretical justification.

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Positions and Intensities of the Absorption Bands of Trypaflavine and Fluorescein Cations

by

B. ŻURAWSKI

Presented by A. JABŁOŃSKI on October 10, 1960

The positions and oscillator strengths of the absorption bands of trypaflavine and fluorescein cations were computed (the oscillator strengths — for some bands only). The chemical formulas of the respective molecules are shown in Fig. 1.

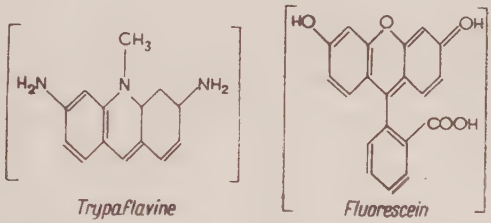

Fig. 1. Chemical formulas of trypaflavine and fluorescein cations

Note: on the left side of the Figure, two first bonds from the right should be double bonds

The cations of either dye present systems of conjugated double bonds. Hence, the FEMO method was used in the computations. Notwithstanding the fact that the cores of the molecules under consideration contain the foreign atoms N and O, the Sommerfeld approximation, i.e., the assumption of constant zero potential along the core was introduced. Moreover, the distances C—C, C—N and C—O

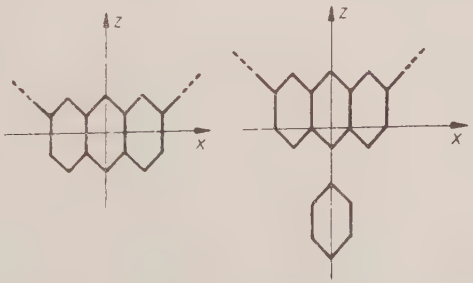

Fig. 2. Cores of trypaflavine and fluorescein molecules

between neighbouring atoms were assumed to be equal and to amount to $D = 1.39$ Å. With the foregoing assumptions, Ruedenberg and Scherr's formalism [1] could be applied.

In the Table, the positions of the absorption bands of either cation are assembled. The oscillator strengths were computed for the bands whose positions had been compared with the experimental data in the papers by Laffitte [2] and Feofilov [3].

TABLE

Positions and oscillator strengths of absorption bands for trypaflavine and fluorescein cations

Molecule	Transition	Transition polarized in direction of axis	$\lambda_{theor.}$ Å	$\lambda_{exp.}$ Å	Oscillator strength (theor.)
Trypaflavine 18 π-electrons	$9 \rightarrow 10$	x	13733		
	$9 \rightarrow 11$	x	5145		
	$9 \rightarrow 12$	z	3983		
	$9 \rightarrow 13$	z	3279		
	$9 \rightarrow 14$	x	2389		
	$9 \rightarrow 15$	x	2202		
	$8 \rightarrow 10$	z	7177		
	$8 \rightarrow 11$	z	3833		
	$8 \rightarrow 12$	x	3148		
	$8 \rightarrow 13$	x	2690	2650 [2]	0.023
	$8 \rightarrow 14$	z	2060		
	$7 \rightarrow 10$	z	4690	4650 [2]	0.000008
	$7 \rightarrow 11$	z	2986	3000 [3]	
	$7 \rightarrow 12$	x	2553		
Fluorescein 24 π-electrons	$10 \rightarrow 11$	x, z	9298		
	$10 \rightarrow 12$	x, z	3980		
	$10 \rightarrow 13$	x, z	3613		
	$10 \rightarrow 14$	x, z	3277		
	$10 \rightarrow 15$	x, z	2428		
	$10 \rightarrow 16$	x, z	2232		
	$10 \rightarrow 17$	x, z	1409		
	$9 \rightarrow 11$	z	4949	4900 [3] 4850 [2]	
	$9 \rightarrow 12$	x	2892		
	$9 \rightarrow 13$	z	2694		
	$9 \rightarrow 14$	x	2502	2500 [2] 2450 [3]	
	$9 \rightarrow 15$	z	1975		
	$8 \rightarrow 11$	x	3368		
	$8 \rightarrow 12$	z	2269	2300 [2]	0.0009
	$8 \rightarrow 13$	x	2145		
	$7 \rightarrow 11$	z	3216		
	$7 \rightarrow 12$	x	2199		
	$6 \rightarrow 11$	x	3071	3000 [2] 2950 [3]	1.75
	$6 \rightarrow 12$	z	2131		

The positions of the absorption bands are seen to be in excellent agreement with the experimental data. It is noteworthy that, in the case of fluorescein, an absorption band at $\lambda = 3000$ Å ($6 \rightarrow 11$) was obtained which, hitherto, had been considered to be an internal band belonging to the substituent group or due to interaction between the substituent and the environment [2]. For trypaflavine, in addition to the absorption bands mentioned by Laffiitte, Feofilov gives one which, here, was found to be accounted for by the transition $7 \rightarrow 11$. The method, moreover, yields absorption bands for which no experimental data are hitherto known.

The oscillator strengths computed are in no very good agreement with the experimental data; thus, for trypaflavine, the band corresponding to the transition $7 \rightarrow 11$ is one of the most intense, whereas computation yielded for it an oscillator strength of 0.000008 (the other results, however, are within reasonable limits). It is to be expected that, by taking into account the corresponding change of the potential on the N and O atoms in the core, good agreement with the experimental data can be obtained.

In the second column of the Table, the symbols x and z denote transitions polarized in the direction of the corresponding axis in Fig. 2.

As distinct from other methods of determining the positions of the absorption bands for trypaflavine and fluorescein, the one utilized in the present investigation involves no parameters in addition to the distance D between neighbouring atoms. It takes account of the fact that the current of π-electrons flows along the entire core, in particular, the current branches at joint points, a circumstance not taken into account in computations carried out hitherto [2].

The author wishes to express his indebtedness to Dr. W. Woźnicki for his valuable advice throughout the present investigation and to Professor A. Jabłoński for suggesting the subject.

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On a Modification of the FEMO Method in the Case of Benzophenone

by

E. JUŹKÓW

Presented by A. JABŁOŃSKI on October 10, 1960

In order to determine the properties of benzophenone, $C_6H_5COC_6H_5$, (Fig. 1), the FEMO method proposed by Ruedenberg and Scherr [1] can be utilized.

The FEMO method is based on the assumption that the properties of molecules presenting a conjugated system of bonds are determined by the electrons moving freely along the core (Fig. 2). The potential along the entire path of the electrons is constant. By Ruedenberg and Scherr's formalism, the path of the electrons should be prolonged by the length of one bond at the free ends. This has been denoted by the dashed line in Fig. 2.

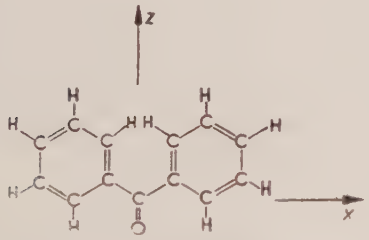

Fig. 1. Formula of benzophenone molecule

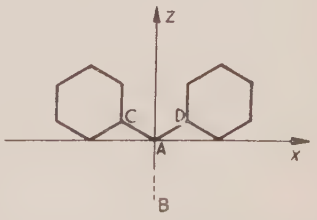

Fig. 2. Core of benzophenone molecule

The results obtained by this method are given in the Table. In the case of benzophenone, the modification of the FEMO method consists in taking account of the fact that the bond between the atoms of the molecule are not equivalent [2], [3]. In the benzophenone molecule, the two benzene rings are linked by a CO group. Assuming the bonds in the benzene rings to present a character intermediate between that of a single and double bond, and taking into account that CO is a strictly double bond, the system of bonds illustrated in Fig. 3 results necessarily. Accordingly, the potential along the molecule core was assumed to possess the form shown in Fig. 4.

Computations were carried out at the fixed value of potential jump $V_2 - V_1 = 3 \times 10^{-12}$ erg. The potential V_2 was varied continuously; the distance A—B (free

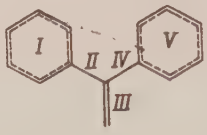

Fig. 3. Bond system in benzophenone molecule

end) was made to assume the four values 1.815 Å, 2.42 Å, 2.8 Å and 3 Å (in the case of constant potential also). The length of the benzene ring was taken to be 6×1.39 Å for both the modified and non-modified FEMO model. The wave numbers

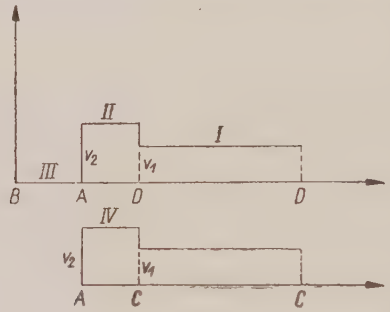

Fig. 4. Form of potential along core of molecule

of the transitions were computed for all foregoing cases and the best agreement with experimental data was obtained for $V_2 = 7.5 \times 10^{-12}$ erg, A—$B = 2.42$ Å.

The Table brings the wave numbers of the transitions and the oscillator strengths for $V_2 = 7.5 \times 10^{-12}$ erg, A—$B = 2.42$ Å together with the experimental values [4] and the results obtained by the non-modified FEMO method.

TABLE

Theoretical as compared with experimental results

Transition between levels	Wave numbers in $cm^{-1} \cdot 10^{-3}$			Oscillator strength		
	exp.	FEMO	FEMO modified	exp.	FEMO	FEMO modified
7 → 8	34.4	29.1	33.0	0.007	0.072	0.012
4,5 → 8	38.8	40.5	38.1	0.42	2.485	1.035
6 → 10,11	49.2	50.2	48.5	0.56	3.854	0.843
4,5 → 10,11	53.3	52.2	52.2	1.14	8.533	3.002

It is noteworthy that, for A—$B = 2.8$ Å and 3 Å, no transitions corresponding to the experimental wave numbers 34400 cm^{-1} and 38800 cm^{-1} were found.

From the Table the modified FEMO method is seen not only to lead to better agreement between the wave numbers computed and their experimental values, but, in the first place, to yield much more satisfactory results for the oscillator strength. This proves that the wave functions obtained by the modified method provide for a much more adequate description of the state of the molecule.

The author wishes to thank Dr W. Woźnicki for his kind help and valuable hints throughout the present investigation.

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The Global Symmetry in Weak Interactions

by

J. LUKIERSKI

Presented by L. INFELD on October 21, 1960

Gamba, Marshak and Okubo [1] have pointed out that the weak interactions might be invariant under the simultaneous interchanges:

$$p \leftrightarrow \nu, \quad n \leftrightarrow e^-, \quad \Lambda \leftrightarrow \mu^-. \tag{1}$$

It is the purpose of this paper to show that the invariance property (1) together with another symmetry

$$p \leftrightarrow \nu, \quad n \leftrightarrow \mu^-, \quad \Lambda \leftrightarrow e^- \tag{2}$$

can be explained with help of the notion of a universal charge N, which describes the weak interactions. This universal charge we shall call neutrino charge.

We assume that the neutrino charge gauge group is realised on four-spinors by means of the Nishijima transformations [2]

$$\psi' = \exp \frac{i}{2}(1 \pm \gamma_5)\, \alpha \psi. \tag{3}$$

This transformation leaves the Dirac equations for leptons

$$(\gamma_\mu \partial_\mu + m_L)\, \psi_L = 0 \qquad L = e, \mu, \nu \tag{4}$$

invariant only if $m_L = 0$. Therefore, the transformation

$$\begin{cases} \psi'_e = \psi_e, \quad \psi'_\mu = \psi_\mu; \\ \psi'_\nu = \exp \dfrac{i}{2}(1+\gamma_5)\, \alpha \psi_\nu \end{cases} \tag{5}$$

leaves all lepton equations (4) invariant. This is the neutrino charge gauge transformation N for leptons.

It is easily seen that transformations (5) together with the assumption of the universality of neutrino charge imply the two-component neutrino of Salam-Landau type [3], [4]. Indeed, because of

$$\exp \frac{i}{2}(1 \pm \gamma_5)\, \alpha = \frac{1}{2}(1 \pm \gamma_5) + \frac{1}{2}(1 \pm \gamma_5)\, e^{i\alpha} \tag{6}$$

we have

(7) $$N\psi_{\nu+} = e^{i\alpha}\psi_{\nu+} \qquad N\psi_{\nu-} = \psi_{\nu-},$$

where

$$\psi_{\nu\pm} = \frac{1}{2}(1\pm\gamma_5)\psi_\nu.$$

If we put

(8) $$\psi_\nu \equiv \psi_{\nu+}$$

we obtain that, for all leptons, neutrino charge N can be expressed with the help of the following formula:

(9) $$N = Q+L,$$

where e^-, μ^-, ν are particles with lepton number $L = 1$.

From the universality of the neutrino charge there now follows precisely limitation (8).

Let us consider only the elementary particles taken into account in (1)*). It is well-known that nucleon can be described with the help of two usual four-spinors ψ_p and ψ_n as well as with the help of Gursey's four-spinors χ and ξ [7]. Electric charge gauge transformation

(10) $$\psi'_p = e^{i\alpha}\psi_p \qquad \psi'_n = \psi_n$$

can be expressed on χ and ξ also with the help of a Nishijima type transformations:

(11) $$\begin{cases} \chi' = \exp\frac{i}{2}(1+\gamma_5)\alpha\chi \\ \xi' = \exp\frac{i}{2}(1-\gamma_5)\alpha\xi. \end{cases}$$

Therefore, we postulate that for baryons and mesons

(12) $$N = Q.$$

This is equivalent to the extension of relation (9) to all elementary particles. By virtue of (12) or (9) we have

(13) $$N\psi_\Lambda = \psi_\Lambda.$$

It follows from our postulate, that the Lagrangian of weak interactions should be determined with help of the following weak source charges only**):

*) This limitation of number of elementary particles is natural if we assume the compound particle model of Markow [5] and Sakata [6] which is based on these six "primary" particles.

**) If the interaction Lagrangian L_t is invariant with respect to the transformation

(14) $$\psi \leftrightarrow \varphi,$$

where ψ and φ posses the same transformation properties with respect to the Lorentz group, ψ and φ correspond to the same value of a certain quantity Q_t, which is the corresponding to L_t source charge.

As an example we may quote the globally symmetric strong interaction Lagrangian, for which the sources are determined by the baryonic charge B, or the electromagnetic interaction Lagrangian, for which the sources are determined by the electric charge.

$$\begin{cases} N(p) = N(\nu) = 1, \\ N(n) = N(\Lambda) = N(e^-) = N(\mu^-) = 0. \end{cases} \tag{15}$$

After taking into account the conventional assignements of universal charges B and L we see that the weak interactions Lagrangian must (due to our hypothesis and universality of B and L) be invariant with respect to the following simultaneous interchanges:

$$\begin{cases} N \leftrightarrow N, \\ B \leftrightarrow L. \end{cases} \tag{16}$$

Because e, μ and n, Λ are described with the help of the same numbers N, B and L, from (16) we obtain, besides (1), the second symmetry (2).

We notice that (1) goes over into (2) after the following interchange:

$$e \leftrightarrow \mu. \tag{17}$$

Thus, the assumption of the weak source charges (15) is equivalent to introduction of the symetries (1) and (17) simultaneously. The test for the choice between our and the original hypothesis of Gamba, Marshak and Okubo should be based on the possibility to distinguish the electron from the muon in the weak interactions (besides the effects connected with different m_e and m_μ).

The neutrino charge hypothesis is usefull for the elimination of the unobservable processes. It is natural to assume that the weak interactions between neutral (with respect to the neutrino charge) fields vanish. Therefore, the unobservability of processes

$$\begin{cases} \mu^\pm \to 2e^\pm + e^\mp \\ \Lambda \to n + e^\pm + \mu^\mp \text{ etc.} \end{cases} \tag{18}$$

follows from our hypothesis as a necessary consequence.

Let us remark finally that the last experimental data on the process

$$\Lambda \to p + e^- + \bar{\nu} \tag{19}$$

are inconsistent with the symmetry (1) [8]. If the renormalization effects are not sufficient to explain these discrepancies, we must change one from the relations (1) as follows:

$$\mu^- \leftrightarrow \Lambda \Rightarrow \mu^- \leftrightarrow \alpha\Lambda + (1 - \alpha)\Sigma^\circ, \tag{20}$$

where $\alpha \sim 0{,}1$.

Because reaction $\Sigma^\circ \to \Lambda + \gamma$ is very fast in comparison to (19), the choice (20) explains the small cross-section of the decay (19).

For $\Lambda \to p + \mu^- + \bar{\nu}$ the experimental results imply $\alpha < 0{,}4$ (8). Thus, if (20) is realised, the invariance property (17) can be maintained.

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K^+-Proton Scattering in a Case of Scalar Coupling

by

M. ŚWIĘCKI and M. SZALEK

Presented by W. RUBINOWICZ on November 8, 1960

The problem of the K^+-nucleon scattering in the Tamm-Dancoff approximation has been considered by Białkowski and Jurewicz [1]. Results obtained are in good agreement with the experimental data but only in the low energy region. In this paper we consider the same problem but for the even parity of K mesons with respect to ΛN of and ΣN pairs.

The interaction Hamiltonian has the form [2]:

$$(1) \qquad \overset{1}{H} = \int d_3x \, [G_\Lambda \psi_N(\vec{x}) \, \psi_\Lambda(\vec{x}) \, \varphi_K(\vec{x}) + G_\Sigma \psi_N(\vec{x}) \, \tau \varphi_K(\vec{x}) \, \psi_\Sigma(\vec{x}) + \text{h.c.}] \, .$$

In order to derive the equation of motion, we use the method given by Królikowski and Rzewuski [3]. In this method the equation of state is:

$$(2) \qquad (E - \overset{\circ}{H}) \, |\Phi_{\| E}\rangle = U(E) | \, \Phi_{\| E}\rangle,$$

where $\overset{\circ}{H}$ denotes the Hamiltonian of free fields, E the total energy of system, and $\overset{1}{H}$ the interaction Hamiltonian. $|\Phi_{\| E}>$ is the projection of the state vector on the subspace containing one K meson and one nucleon. In the lowest approximation $U(E)$ is given by

$$(3) \qquad U(E) = P_\| \overset{1}{H} P_\perp \frac{P}{E - \overset{\circ}{H}} P_\perp \overset{1}{H} P_\|,$$

where $P_\| = |\Phi_{\| E}\rangle \langle \Phi_{\| E}|$; $P_\perp = 1 - P_\|$.

In the usual way, we obtain in the center of the mass system the integral equation for the scattering amplitude $f^{\alpha\beta}(\vec{p})$. In this equation we omit the self energy and vacuum parts, [1]. The resulting equation is

$$(4) \qquad [E - E(p) - \Omega(p)] \, f^{\alpha\beta}(\vec{p}) = \frac{G_\Lambda^2}{16\pi^3} \int d_3 p' \, K_\Lambda(\vec{p}, \vec{p}') \, f^{\beta\alpha}(\vec{p}') +$$

$$+ \frac{G_\Sigma^2}{16\pi^3} \tau_\mu^{\beta\nu} \tau_\mu^{\alpha\varrho} \int d_3 p' \, K_\Sigma(\vec{p}, \vec{p}') \, f^{\nu\varrho}(\vec{p}') \, ;$$

$$K_{\Lambda}(\vec{p},\vec{p}') = \frac{Mm\,\Lambda_{+}(\vec{p})}{[\Omega(\vec{p})\,\Omega(p')\,E(p)\,E(p')]^{1/2}\,\varepsilon(\vec{p}+\vec{p}')} \times$$

$$\times\left\{\frac{\Pi_{+}(\vec{p}+\vec{p}')}{E-\varepsilon(\vec{p}+\vec{p}')-\Omega(p)-\Omega(p')}+\frac{\Pi_{-}(-\vec{p}-\vec{p}')}{E-\varepsilon(\vec{p}+\vec{p}')-E(p)-E(p')}\right\}$$

$$\Lambda_{\pm}(\vec{p}) = \frac{m\pm\hat{p}}{2m}, \qquad \Pi_{\pm}(\vec{p}) = \frac{M\pm\hat{p}}{2M},$$

m, $E(p)$; μ, $\Omega(p)$; M, $\varepsilon(p)$; $\mathfrak{M}$, $\in(p)$ are the masses and energies of N, K, Λ and Σ, respectively. One can obtain $K_{\Sigma}(\vec{p},\vec{p}')$ from $K_{\Lambda}(\vec{p},\vec{p}')$ by replacing the mass of Λ by the mass of Σ. We consider all the constants in Eq. (4) as renormalized ones.

The separation of angular variables in Eq. (4) has been performed in the same manner as in [4], and the separation of isospin states as, e.g., in [1]. We consider only the following process

$$K^{+}+p \to K^{+}+p$$

and the $S_{1/2}$ wave. The $P_{1/2}$ and $P_{3/2}$ waves are discussed in the low energy approximation.

After the separation of angular variables and isospin states we introduce the non-singular amplitude of the form

$$f(p) = \delta(E-E(p)-\Omega(p)) + P\frac{\varphi(p)}{E-E(p)-\Omega(p)}, \tag{5}$$

where $f(p)$ is the scattering amplitude for the $T=1$ state with definite total angular momentum and parity. The non-singular amplitude $\varphi(p)$ is connected with the phase shift as follows:

$$\varphi(p_0) = -\frac{1}{\pi}\operatorname{tg}\delta, \tag{6}$$

where p_0 is the momentum on the energy shell.

Finally we get the integral equation for the non-singular amplitude:

$$\varphi(p) = \varphi_B(p) + P\int_0^{\infty} E-E(p')-\Omega(p')\left[\frac{G_{\Lambda}^2}{16\pi^2}K_{\Lambda}(p,p')+\frac{G_{\Sigma}^2}{16\pi^2}K_{\Sigma}(p,p')\right]\varphi(p')$$

$$\varphi_B(p) = \frac{E(p_0)\,\Omega(p_0)}{p_0 E}\left[\frac{G_{\Lambda}^2}{16\pi^2}K_{\Lambda}(p,p_0)+\frac{G_{\Sigma}^2}{16\pi^2}K_{\Sigma}(p,p_0)\right]$$

$$K_{\Lambda}(p,p') = \frac{p'^2}{[\Omega(p)\,\Omega(p')\,E(p)\,E(p')]^{1/2}}\Big\{[E(p)+m]\,[(E-2m-M)\,N_{\alpha}(B)+ \tag{7}$$

$$-(E+2m+M-E(p)-E(p')-\Omega(p)-\Omega(p'))\,N_{\alpha}(C)]+$$

$$+\frac{pp'}{E(p')+m}[(E+2m+M)\,N_{\beta}(B)+$$

$$-(E-2m-M-E(p)-E(p')-\Omega(p)-\Omega(p'))\,N_{\beta}(C)]\Big\}.$$

Here E denotes the total energy of the system, and $C = E - \Omega(p) - \Omega(p')$, $B = E - E(p) - E(p')$. $K_\Sigma(p, p')$ may be obtained from $K_\Lambda(p, p')$ by replacing M by $\mathfrak{M}$ everywhere. The functions N_α and N_β for the $S_{1/2}$, $P_{1/2}$ and $P_{3/2}$ states are given in [1]. $\varphi_B(p)$ describes the scattering process in the first Born approximation.

The kernel of integral equation is a singular one. For energies $E < M + 2\mu$ we have only to deal with the singularity which derives from the denominator $E(p') + \Omega(p') - E$. The integral equation with the singular kernel has been solved by the seminumerical method. This method is just the same as in [1] and [4]. One gets, as result of approximation, the degenerate kernel. The integral equation obtained in this manner is easy to solve. In our calculations we have put the masses of Λ and Σ both numerically equal to the mass of Λ. In this case the only parameter still undetermined is the sum of the coupling constants $\frac{G_\Lambda^2 + G_\Sigma^2}{4\pi}$.

In the lowest energies the contribution from the $P_{1/2}$ and $P_{3/2}$ waves is practically of no importance (this results from the numerical calculations). On account of this we consider the $S_{1/2}$ wave only. The results of calculations do not agree with the experimental data. By adjusting the results of calculations for $S_{1/2}$ wave in the low energy ($E_L = 65$ MeV in the laboratory system) to the experiment we have obtained the value of the sum of coupling constants equal to 1,4 (see Fig. 2). For this value the results are as follows: The $S_{1/2}$ phase shifts are positive, whereas the $P_{1/2}$ and $P_{3/2}$ ones are negative — thus the interaction in the lowest energies is attractive. From Fig. 2 we see also that for any value of $g^2 = \frac{G_\Lambda^2 + G_\Sigma^2}{4\pi}$ the agreement with the experimental data cannot be obtained (the experimental data point to the slow increase of the cross-section). The contribution from the $P_{1/2}$ and $P_{3/2}$

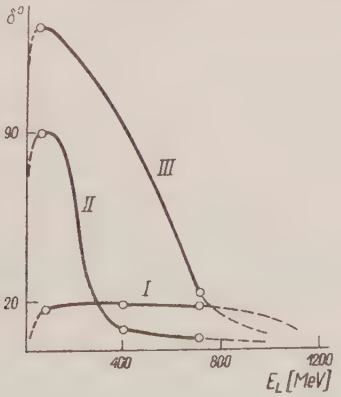

Fig. 1. Calculated $S_{1/2}$ phase shifts for $K^+ p$ scattering
I — $g^2 = 1{,}4$; II — $g^2 = 4{,}9$; III — $g^2 = 21$

waves in higher energies will not improve this situation. This can be observed by comparison with the paper by Białkowski and Jurewicz: The $S_{1/2}$, $P_{1/2}$ and $P_{3/2}$ kernels of integral equations have similar values in both the papers but different signs; their behaviour in the neighbourhood of zero and infinity is the same. Thus,

from this paper and from [1] it follows that coupling between barions and mesons is rather a pseudoscalar one.

Finally the possibility of appearance of $S_{1/2}$ resonance has been considered. From Fig. 1 we see that the resonance occurs: for $g^2 = 4{,}9$ and for $g^2 = 21$. On account of the decreasing of the $S_{1/2}$ phase shift to zero for the energy $E_L \sim 1{,}2$ BeV in the laboratory system (this results from the numerical calculations) we get also the "pseudo-resonance" besides the resonance (see Fig. 1).

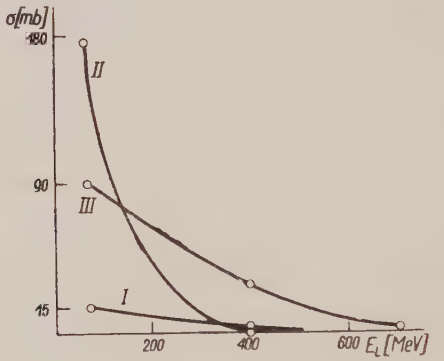

Fig. 2. Calculated $S_{1/2}$ total cross-sections for $K^+ p$ scattering
I — $g^2 = 1.4$; II — $g^2 = 4.9$; III — $g^2 = 21$

The sources of errors and their discussion are given in reference [1]. There is little probability that the errors could change the general results of our paper.

The authors are much indebted to Professor W. Królikowski and to Dr. G. Białkowski and Dr. A. Jurewicz for their kind assistance, interest and valuable discussions.

Summary

The problem of the K^+-proton scattering in the Tamm-Dancoff approximation is considered. The coupling between K mesons and barions is assumed to be a scalar one. The $S_{1/2}$ phase shift and the total cross-section are presented. The $P_{1/2}$ and $P_{3/2}$ waves are discussed in the low-energy approximation. The possibilities of appearance of $S_{1/2}$ resonance are considered. The results obtained do not agree with the experimental data, suggesting that the coupling is not a scalar one.

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PHYSIQUE EXPÉRIMENTALE

Effet de la concentration de l'activateur sur la profondeur des pièges à électrons dans le ZnS

par

U. SZMYT et A. WRZESIŃSKA

Présenté par A. JABŁOŃSKI le 27 juillet 1960

Introduction

Au cours de recherches sur les propriétés électriques des semi-conducteurs d'impureté et en premier lieu sur leur conductivité, de nombreux auteurs ont constaté que l'énergie d'activation de la composante à basse température diminue à mesure que la concentration des impuretés augmente.

Dans les semi-conducteurs d'impureté, la conductivité électrique dépend de la température conformément à la formule

$$\delta(T) = A_1 \exp(-E_1/kT) + A_2 \exp(-E_2/kT), \tag{1}$$

où A_1 et A_2 designent les constantes, E_1 et E_2 — énergies d'activation, k — constante de Boltzmann, et T — température absolue (cf. p.ex. [1]).

L'énergie d'activation plus élevée E_2 correspond à la formation de charges libres du courant à la suite des transitions bande-bande, tandis que l'énergie plus faible E_1 correspond au processus de la libération des charges de niveaux d'impureté soit du donneur soit de l'accepteur, selon le type de la conductivité. E_2 dépasse de beaucoup E_1, et le premier terme dans la formule (1) prédomine dans la région de basses températures. Anderson et Morton [1], Busch [2], Pearson et Barden [3] ainsi que d'autres ont démontré que l'énergie d'activation E_1, étant la différence entre l'énergie du niveau du donneur et celle de la limite inférieure de la bande de conductivité, où bien entre l'énergie du niveau de l'accepteur et celle de la limite supérieure de la bande occupée la plus proche, baisse à mesure que la concentration augmente. E_2 — énergie d'activation de la composante de la conductivité à haute température — est indépendante de la concentration des impuretés. Les recherches d'Andrews et Hogarth [4] sur la force thermoélectrique ont établi des changements analogues de l'énergie d'activation (d'ionisation) des impuretés.

Résultats

Pour le phénomène de phosphorescence des luminophores photoconducteurs, les niveaux locaux situés au-dessous de la bande de conductivité — nommés pièges à électrons — jouent un rôle essentiel. Dans l'état non-excité du phosphore, ceux-ci n'étant pas occupés, diffèrent des niveaux des donneurs; toutefois, lorsque le phosphore est excité, ils sont plus ou moins occupés par des électrons et deviennent des niveaux donneurs. L'excitation coupée, ils deviennent une source des électrons de conductivité. Dans le présent travail nous essayons d'établir, si — oui ou non — la profondeur thermique des pièges depend de la concentration de l'activateur.

La profondeur thermique peut être déterminée en étudiant les courbes d'extinction ou bien celles de la thermoluminescence. C'est la dernière méthode que les auteurs ont choisie.

Les courbes de thermoluminescence furent tracées en utilisant un dispositif dont la construction est présentée dans le travail de A. Wrzesińska [5]. La profondeur des pièges fut calculée en se basant sur les positions des maxima des courbes d'après la méthode donnée dans [5]. Les mesures furent effectuées avec une série de phosphores à base ZnS activés au cuivre. Dans les phosphores activés à base ZnS, outre les pièges profonds, dûs à l'activateur, on trouve d'autres, moins profonds, qui n'en dependent pas. Les profondeurs de ces derniers se trouvent entre 0,23 et 0,40 eV.

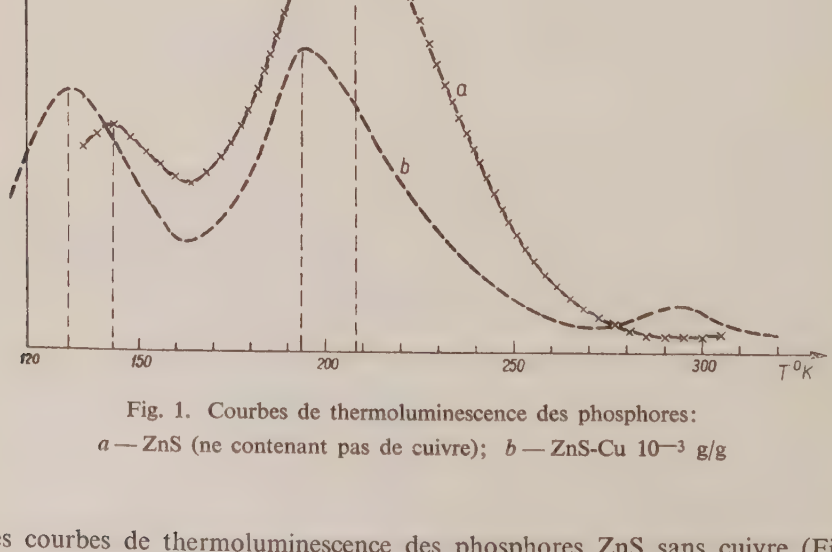

Fig. 1. Courbes de thermoluminescence des phosphores:
a — ZnS (ne contenant pas de cuivre); *b* — ZnS-Cu 10^{-3} g/g

Les courbes de thermoluminescence des phosphores ZnS sans cuivre (Fig. 1, courbe *a*) et de ceux où le cuivre est présent (courbe *b*) démontrent que dans les conditions de notre experience un seul „ glow peak" est dû à la présence du cuivre, notamment le „peak" qui correspond à la température la plus élevée sur la courbe *b*.

Il était à prévoir qu'on pourrait modifier la concentration des pièges dûs à la présence de l'activateur en faisant varier la concentration de celui-ci. Les auteurs ont donc préparé une série de phosphores ZnS activés au cuivre, en variant la concentration de l'activateur entre 0 et 10^{-3} g/g. La Fig. 2 présente les courbes de

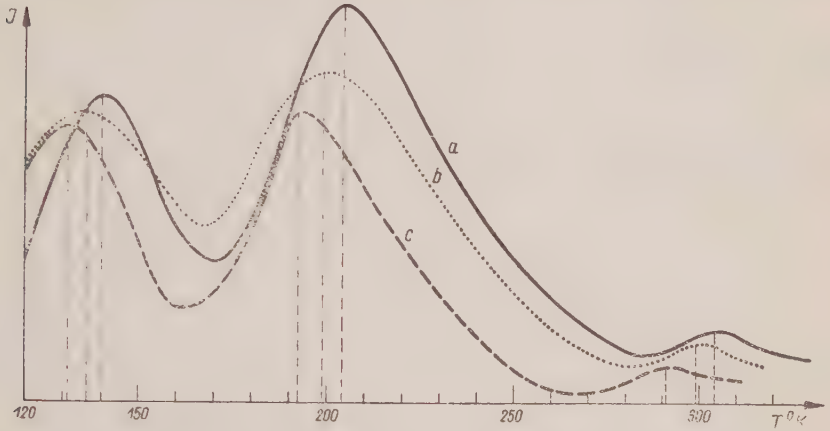

Fig. 2. Courbes de thermoluminescence des phosphores à différentes concentrations de l'activateur: *a* — ZnS-Cu 10^{-5} g/g; *b* — ZnS-Cu 10^{-4} g/g; *c* — ZnS-Cu 10^{-3} g/g

thermoluminescence pour les trois concentrations suivantes: courbe *a*, phosphore ZnS-Cu 10^{-5} g/g; courbe *b*, ZnS-Cu 10^{-4} g/g; courbe *c*, ZnS-Cu 10^{-3} g/g. Ces courbes présentent un déplacement systématique des „glow peaks" vers les températures basses à mesure que la concentration de l'activateur augmente. C'est en Fig. 1 que l'on constate la différence la plus marquée entre les positions des

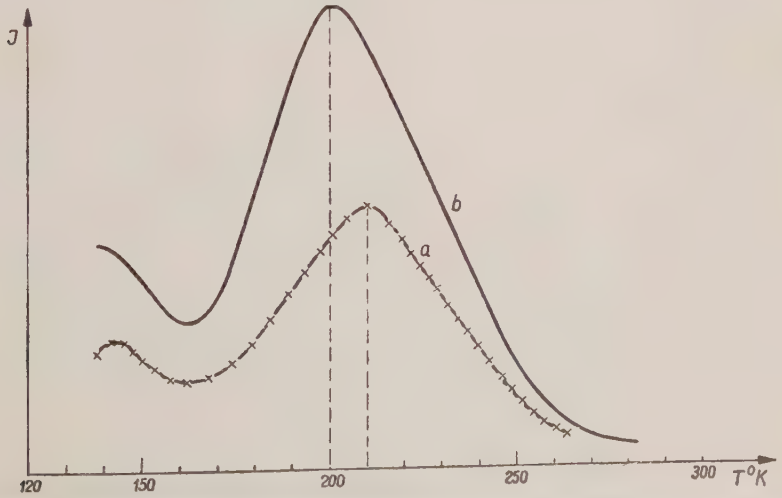

Fig. 3. Courbes de thermoluminescence d'un phosphore ZnS sans Cu à différents degrés d'excitation: *a* — excitation faible; *b* — excitation forte

maxima correspondants, où la courbe *a* est celle du phosphore ZnS ne contenant pas de cuivre, tandis que la courbe *b* correspond à la concentration maximum de Cu 10^{-3} g/g. Bien que les déplacements (même lorsqu'il s'agit de différences importantes de concentration) ne soient pas très grands, ils dépassent nettement l'erreur expérimentale.

Les températures correspondant aux „glow peaks", ainsi que la profondeur des pièges sont présentés sur la Table.

TABLE

Phosphore	ZnS — Cu 10^{-3}	ZnS — Cu 10^{-4}	ZnS — Cu 10^{-5}	ZnS
T° K	131 193 291	136 198 298	139 204 303	143 210
Profondeur des pièges, en eV	0,28 0,42 0,65	0,29 0,43 0,66	0,30 0,45 0,68	0,31 0,46

Il est toutefois intéressant que ce ne sont pas seulement les maxima correspondant aux pièges dûs à la présence de l'activateur qui subissent un déplacement, mais aussi les maxima de basse température, correspondant aux pièges dans le phosphore à ZnS sans métal étranger, ces déplacements étant à peu près égaux pour les pièges des toutes sortes.

Discussion

On trouve dans les travaux publiés plusieures tentatives d'interprétation de l'effet de la concentration des impuretés sur la profondeur des niveaux des donneurs et des accepteurs. On supposait d'abord que cet effet provenait d'une superposition plus ou moins accentuée des fonctions d'onde des niveaux d'impureté. Toutefois, les calculs basant sur cette hypothèse démontrèrent que l'effet de superposition des fonctions d'onde des impuretés dans le silicium est beaucoup plus failble que l'effet constaté expérimentalement. Pearson et Barden sont d'avis que le changement de l'énergie d'ionisation des niveaux d'impureté est dû en premier lieu à la variation de la concentration des charges libres du courant. Selon ces auteurs l'abaissement de l'énergie d'ionisation à mesure de l'accroissement de la concentration résulterait de l'attraction coulombienne entre les charges libres du courant et le centre ionisé. Dans les semi-conducteurs d'impureté, dans la région des températures peu élevées la concentration de ces charges est proportionnelle à celle des impuretés ionisées. Ils ont proposé la formule suivante pour la variation de l'énergie d'ionisation E en fonction de la concentration N:

$$E = E_0 - a \cdot N^{1/3}, \tag{2}$$

où a — constante du matériel, et E_0 — énergie d'ionisation à concentration infiniment petite. Les résultats des expériences de Pearson et Barden concordent avec la formule donnée. Cependant, la thèse de Pearson et Barden provoqua une critique

de la part de Castellan et Seitz [7] puisque, d'après cette interprétation, N dans la formule (2) doit désigner la concentration des impuretés ionisées, tandis que la concordance avec les résultats expérimentaux fut obtenu en substituant pour N la concentration de toutes les impuretés. Dans la série de mesures effectuées par Pearson et Barden, la fraction des impuretés ionisées variait fortement.

Les calculs de Pincherle [6], basés sur l'hypothèse de Pearson et Barden quant à l'effet d'écran exercé par les charges libres, ont abouti, pour le silicium du type n, à des variations de l'énergie d'activation avec la concentration qui sont un peu plus petites que celles qui résultent des mesures. Pourtant, cette divergence peut être attribuée au fait que Pincherle basait ses calculs sur l'hypothèse d'une distribution homogène des électrons de conductivité, tandis qu'en réalité leur densité augmente dans le voisinage de l'ion positif.

Les résultats du présent travail semblent indiquer aussi que la cause immédiate des variations des profondeurs des pièges dans les phosphores ZnS-Cu réside dans la variation de la concentration des charges libres. L'accroissement de la concentration de l'activateur entraîne une augmentation de celle des centres luminogènes. Donc, avec le même degré d'excitation (le temps et l'intensité de la lumière excitante étant les mêmes) on obtient une plus grande concentration des électrons libres aussi bien au cours de l'excitation, qu'au cours du processus d'extinction. Il faut tenir compte, il est vrai, de la complication relevant du fait que, dans le ZnS-Cu, nous avons deux espèces de centres luminogènes, notamment, des centres „bleus" dans le phosphore ZnS sans Cu, et des centres „verts" dûs à l'introduction du cuivre. Ce ne fut que la concentration de ces derniers que les auteurs faisaient varier dans leurs expériences.

Si l'hypothèse de l'effet d'écran exercé par les charges libres est valable, des variations de l'énergie d'ionisation (variations de profondeur) des pièges devraient apparaître également au cours des variations du degré d'excitation. La Fig. 3 montre l'effet attendu: la courbe de thermoluminescence a y correspond à une excitation plus faible et la courbe b — à une excitation plus forte du phosphore ZnS sans cuivre.

Les variations de la profondeur des pièges avec la concentration de l'activateur rélatées dans le présent travail sont très petites par rapport à celles des niveaux d'impureté trouvées au cours des recherches sur les propriétés électriques des semiconducteurs. Une raison plausible résiderait dans la constante diélectrique relativement basse pour ZnS. La constante diélectrique effective du ZnS est d'environ 7, tandis que, dans le cas du silicium, cette constante atteint environ 13. On a constaté des variations beaucoup plus grandes de la profondeur des niveaux locaux en fonction de leur densité pour le TiO_2, dont la constante diélectrique est $\sim$ 80 (cf., p.ex. [9]). Les variations de la profondeur des pièges en fonction de la concentration de l'activateur résulteraient aussi de la théorie dite multimoléculaire de Wise [8]. Toutefois, ceci ne concernerait que les pièges dûs aux centres luminogènes. Une interprétation de nos résultats expérimentaux basée sur cette théorie devrait admettre que les deux groupes de pièges observés à basse température sont dûs aux centres bleus. D'autres auteurs ont cependant démontré que les pièges d'une profondeur

d'environ 0,4 eV sont des pièges de surface, tandis que ceux d'environ 0,28 eV sont considérés comme étant dûs aux trous laissés par les ions négatifs.

En ce qui concerne l'effet du degré d'excitation sur la position du „glow peak", on peut interpréter le déplacement expérimental non comme une variation de la profondeur des pièges, mais aussi comme preuve que la thermoluminescence dans les phosphores ZnS appartient, dans une grande mesure, au type bimoléculaire. Dans le cas d'un mécanisme bimoléculaire, l'abaissement du degré d'excitation du phosphore entraîne un déplacement du „glow peak" vers des températures plus élevées. Si l'on admet, toutefois, une cinétique du type monomoléculaire dans les phenomènes impliquant des éléctrons thermiques (extinction de phosphorescence et de thermoluminescence, [10]), alors le déplacement obtenu dans le présent travail fournit une preuve de l'effet de la concentration des électrons de conductivité sur la profondeur des pièges. L'hypothèse à la base des considérations précédentes est celle qu'un seul „glow peak" correspond à une valeur discrète de la profondeur du piège. Les courbes de thermoluminescence présentées ici ont été obtenues avec un accroissement de température de 0,4 degré par seconde. Ceci représente une valeur plutôt élevée, ce qui a eu pour suite que certains „glow peaks" n'ont pas été séparés et qu'une partie des pièges n'a pas pu être isolée. Les auteurs se proposent de poursuivre leurs recherches sur l'effet de la concentration sous des conditions de changements de temperature moins rapides. Il paraît peu probable toutefois que ceci puisse conduire à des résultats essentiellement différents de ceux qui viennent d'être présentés.

Les auteurs remercient M. le Professeur A. Jabloński pour les discussions et l'interêt qu'il a bien voulu porter à leur ouvrage.

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EXPERIMENTALE PHYSIK

Über die Konzentrationslöschung der Fluoreszenz von Anthracen in Plexiglas

von

A. KAWSKI und Z. POLACKI

Vorgelegt von A. JABŁOŃSKI am 21 Oktober 1960

In früheren Veröffentlichungen [1], [2] wurde über die experimentellen Untersuchungen von Konzentrationsdepolarisation für einige feste Fluoreszenzlösungen berichtet. Es hat sich gezeigt, dass für kleine Konzentrationen von Anthracen, Rhodamin B und „in Wasser lösbares gelbliches Eosin" in Plexiglas, die Messergebnisse sehr gut mit der Jabłońskischen Theorie der Selbstdepolarisation der Photolumineszenz von Lösungen [3], [4] übereinstimmen. Für grosse Konzentrationen von Anthracen in Plexiglas, dagegen, wurde eine Abweichung der Versuchsergebnisse von der von Jabłoński hergeleiteten Formel (1) beobachtet.

Die Theorie von Jabłoński [3], die für unendlich grosse Viskosität ausgearbeitet wurde, liefert für den Fall, wenn keine Konzentrationslöschung vorliegt, folgende Beziehung:*)

$$\frac{\frac{r_0}{r} - \frac{1}{3}}{\frac{1}{P_0} - \frac{1}{3}} = \frac{v^2}{2(v - 1 + e^{-v})}, \tag{1}$$

wobei: r — die Emissionsanisotropie, P — der Polarisationsgrad, $v = v \cdot n$, v — eine für die Wahrscheinlichkeit der Energiewanderung charakteristische von n unabhängige Konstante, n — die Zahl der Moleküle in 1 cm^3 bedeuten. $r \to r_0$ für $v \to 0$ und $P \to P_0$ für $v \to 0$.

Die Gleichung (1) nimmt für sehr kleine Werte von n ($v \ll 1$) die folgende Form an

$$\frac{r_0}{r} = 1 + \frac{1}{3} v. \tag{2}$$

*) Die Emissionsanisotropie für den Fall der polarisierten Erregung der Fluoreszenz beträgt [5]

$$r = \frac{2P}{3 - P}.$$

In der früheren Arbeit vermutete einer der Autoren (A.K.) dass die Abweichung der gemessenen Punkte von der theoretischen Kurve (1) durch die Vernachlässigung der Selbstlöschung in Gl. (1) verursacht ist. Um diese Vermutung zu prüfen, haben wir die Messungen der relativen Fluoreszenzausbeute des Anthracens in Plexiglas ausgeführt, wovon wir in der vorliegenden Arbeit berichten. Die Messung der Ausbeute erfolgte mit Hilfe einer photoelektrischen Anordnung [7]. Das erregende Licht, von einer Quecksilberhöchstdrucklampe emittiert, wurde durch einen Woodfilter filtriert. Unter einem kleinen Winkel zu der Beobachtungsrichtung fallend, erregte es die Fluoreszenz einer dünnen Luminophorplatte. Nach den Messungen der relativen Fluoreszenzintensitäten für alle Konzentrationen des Anthracens in Plexiglas und nach der vorhergehenden Bestimmung des Extinktionsmoduls des erregenden und des Fluoreszenzlichtes wurde die relative Fluoreszenzausbeute

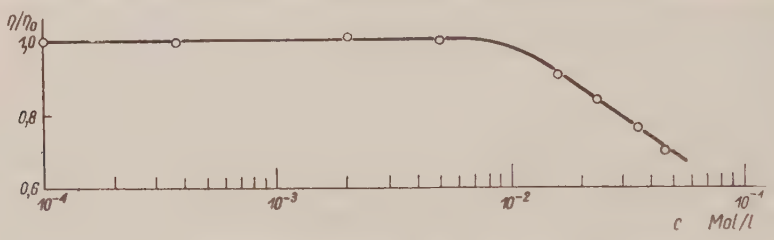

Abb. 1. Konzentrationslöschung von Anthracen in Plexiglas

auf Grund der in der Försterschen Monographie [8] gegebenen Formeln berechnet. Bei der Berechnung der Fluoreszenzausbeute wurde auch die Korrektur für die Polarisationsgradänderung mit der Konzentration in Betracht genommen. Bei grossen Konzentrationen wurde eine Oberflächenfluoreszenz beobachtet, welche von einer Sekundärfluoreszenz von geringer Intensität begleitet war. Daher konnte für diese Konzentrationen die relative Fluoreszenzausbeute berechnet werden, bei der Bedingung allerdings, dass eine vollständige Absorption des erregenden Lichtes erfolgt. Bei vollständiger Absorption des erregenden Lichtes und Beobachtung durch die Eintrittswand ist die Fluoreszenzintensität der inneren Fluoreszenzausbeute proportional [8].

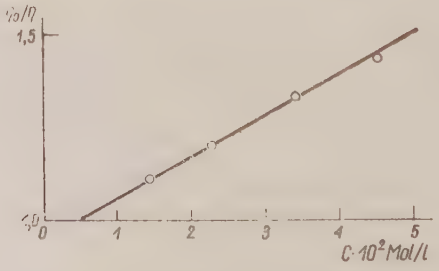

Abb. 2. Konzentrationslöschung von Anthracen in Plexiglas

In Abb. 1 und 2 ist die erhaltene Konzentrationslöschkurve von Anthracen in Plexiglas wiedergegeben. Bemerkenswert ist die geringe Löschung der Fluoreszenz von Anthracen in Plexiglas (hohe Halbwertskonzentration), ähnlich wie bei Anthracen in Paraffinöl [8].

In [1] wurden die Messergebnisse von Polarisationsgraden in Abhängigkeit von der Konzentration des Anthracens in Plexiglas mit der Jabłońskischen theoretischen Kurve (1) für einen mittleren Radius der Wirkungssphäre von $R = 44$ Å verglichen.

Aus unseren jetzigen Messungen von Fluoreszenzausbeuten geht es hervor, dass die Konzentrationslöschung schon bei Konzentrationen $\sim 10^{-2}$ Mol/l beginnt. Aus diesem Grunde kann man nicht einen mittleren Radius der Wirkungssphäre für alle Konzentrationen annehmen, sondern man muss die Messpunkte von kleinen Konzentrationen bevorzugen. In diesem Falle beträgt der mittlere Radius der Wirkungssphäre $R = 45$ Å.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК
СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ
И ФИЗИЧЕСКИХ НАУК

1960

ТОМ VIII Резюме статей ВЫПУСК 11-12

Д. ПШЕВОРСКАЯ-РОЛЕВИЧ, **ОБ ИНВОЛЮЦИЯХ n-ПОРЯДКА**

стр. 735—739

В настоящей работе обобщается несколько теорем из работы [2], касающихся некоторых свойств инволюции порядка 2 и решений инволюционного уравнения порядка 2 с постоянными коэффициентами для инволюций произвольного натурального порядка.

Д. ПШЕВОРСКАЯ-РОЛЕВИЧ, **ОБ ИНВОЛЮЦИОННЫХ УРАВНЕНИЯХ n-ПОРЯДКА** . . . . . стр. 741—746

В работе рассматриваются уравнения вида:

$$(1) \qquad K_A x = (A_0 + A_1 S + \ldots + A_{n-1} S^{n-1} + T_A) x = x_0,$$

где x и x_0 — элементы линейного пространства X, $A_i (i = 0, 1, \ldots, n-1)$ — алгебраически линейные операции, преобразующие пространство X на себя и коммутативные между собой; S — инволюция n-порядка, т.е. операция алгебраически линейная такая, что $S^n = I$ (где I обозначает тождественность), T_A — некоторая линейная операция, пока точнее неопределенная.

Доказывается, что если операционный циклический определитель D_A, образовавшийся из операции A_i обратим, а также $T_A \in J_i$ и $SA_i - A_i S \in J$, где J является некоторым идеалом операции, то всегда существует такая операция R_A (зависящая лишь от A_i и S), что

$$R_A K_A = I + T'_R; \qquad K_A R_A = I + T''_R$$

а также T'_R и T''_R принадлежат к идеалу J.

Из этого утверждения легко вытекают условия разрешимости уравнения (1), аналогичные теоремам Фредгольма.

З. ФРОЛИК, **О ТОПОЛОГИЧЕСКОМ ПРОИЗВЕДЕНИИ ПАРАКОМПАКТНЫХ ПРОСТРАНСТВ** . . . . . стр. 747—750

Вполне регулярное топологическое пространство P называем полным пространством (в смысле Чеха), если оно является множеством G_δ в своей компактификации Стона-Чеха $\beta(P)$.

Доказываются следующие две теоремы:

Теорема 1. *Топологическое произведение счетного количества паракомпактных и полных пространств является паракомпактным и полным пространством.*

Теорема 2. *Топологическое произведение метрического пространства и параметрического полного пространства является паракомпактным и полным пространством.*

Кроме того, даются следующие два равносильные условия для вполне регулярного пространства P паракомпактности (Теорема 4):

a) если P является вполне регулярным пространством, $P \subset R$ и $\overline{P} = R$, то существует пространство S паракомпактное и S полное такое, что $P \subset S \subset R$;

b) если R является вполное регулярным и локально компактным пространством, $P \subset R$ и $\overline{P} = R$, то существует пространство S паракомпактное и S полное такое, что $P \subset S \subset R$.

В. ПОГОЖЕЛЬСКИЙ, **О РЕГУЛЯРНОЙ НЕПРЕРЫВНОСТИ ПРОИЗВОДНЫХ НАИБОЛЬШЕГО РЯДА ОБОБЩЕННЫХ ИНТЕГРАЛОВ ПУАССОНА-ВЕЙЕРШТРАССА ОТНОСИТЕЛЬНО ПАРАБОЛИЧЕСКОЙ СИСТЕМЫ** стр. 751—755

Автор доказывает, что производные M-го ряда обобщенных интегралов Пуассона-Вейерштрасса, определенных формулами (6) выполняют обобщенное условие Гельдера (9) в целом пространстве E. Предполагается, что коэффициенты параболической системы (1) удовлетворяют условию Гельдера и что функции $f_\beta(Y)$ интегрируемы в каждой измеримой и ограниченной части пространства E и выполняют неравенство (7).

Ч. БЕССАГА и А. ПЕЛЧИНСКИЙ, **О ГОМЕОМОРФИЗМЕ НЕКОТОРЫХ ПРОСТРАНСТВ БАНАХА** . . . . . стр. 757—761

В работе доказано, что всякое сепарабельное пространство Банаха, содержащее замкнутое линейное подпространство, гомеоморфное пространству l, само гомеоморфно l. Как следствие из этого факта и из ранее известных теорем Кадеца и Клии ([8], [9], [11], [16] можно получить, что все известные сепарабельные бесконечномерные про-

странства Банаха гомеоморфны между собой. В частности, все пространства $C(Q)$ — всех непрерывных функций, определенных на бесконечных метрических компактах, все бесконечномерные подпространства любого пространства с безусловным базисом, все бесконечномерные подпространства пространства L — гомеоморфны пространству l.

Б. ГЛЕЙХГЕВИХТ, **АЛГЕБРЫ С КВАЗИИНВОЛЮЦИЕЙ** . стр. 763—765

В работе рассматривается класс неассоциативных алгебр, в которых определена операция квазиинволюции. Формулируется теорема о возможности построения этих алгебр с помощью задания соответствующей операции умножения в ассоциативных алгебрах с инволюцией, а также теорема о точном их представлении. Частным случаем алгебр с квазиинволюцией являются τ-алгебры, аналогичные рассматриваемым раньше автором τ-кольцам. Устанавливается необходимое и достаточное условие для того, чтобы данная алгебра с квазиинволюцией была τ-алгеброй и теорема о возможности погружения каждой алгебры с квазиинволюцией в τ-алгебре.

В. КАРЧЕВСКИЙ, **ПРИБЛИЖЕННЫЕ ФОРМУЛЫ ДЛЯ ДИФФРАКЦИОННОЙ ЭЛЕКТРО-МАГНИТНОЙ ВОЛНЫ.** II. . . . . стр. 767—772

Приближенные формулы, полученные в I-ой части работы продискутированы в области, находящейся вблизи границы геометрической тени, а следовательно в области наиболее существенной с точки зрения физики. Оказалось, что в хорошем приближении интенсивность диффракционной электро-магнитной волны в вышеупомянутой области согласуется с результатами скалярной теории Кирхгоффа, а следовательно, тем самым, и с экспериментом. Наибольший вклад в значения векторов $\vec{E}$ и $\vec{H}$ вносят типовые интегралы Френеля. Полученные результаты являются правильными для произвольной пространственной ориентировки осцилирующего электрического либо магнитного диполя и в широких пределах не зависят от формы диффракционного экрана.

Г. СТАХОВЯК, **МЕТОД РЕШЕНИЯ УРАВНЕНИЙ ПОТЕНЦИАЛА И ТЕМПЕРАТУРЫ ДЛЯ ПРОВОДНИКОВ В МАГНИТНОМ ПОЛЕ И ПРОБЛЕМА ЕДИНСТВЕННОСТИ** . . . . . стр. 775—775

Разработан метод решения системы уравнений потенциала и температуры для проводников в магнитном поле при следующих граничных условиях: проводник электрически изолирован, температура задана на его поверхности.

Доказана единственность решения этой проблемы.

Я. ЖЕВУСКИЙ, **ОДНОЧАСТИЧНЫЕ ФУНКЦИИ ГРИНА В СПИНОРНОМ ПРОСТРАНСТВЕ** . . . . . стр. 777—782

Исследованы решения инвариантных дифференциальных уравнений в спинорном пространстве с точки зрения их связи с решениями проблемы одной частицы в квантовой теории поля. Оказалось, что множество решений линейных уравнений в спинорном пространстве содержит функции, которые связаны с решениями нелинейных уравнений в пространстве Минковского. Этот факт указывает на возможность нового метода формулировки проблемы взаимодействия.

Я. ЖЕВУСКИЙ, **ДВУЧАСТИЧНЫЕ ФУНКЦИИ ГРИНА В СПИНОРНОМ ПРОСТРАНСТВЕ** . . . . . стр. 783—787

Исследованы решения инвариантной системы дифференциальных уравнений второго порядка в спинорном пространстве с точки зрения их связи с решениями проблемы двух частиц в квантовой теории поля. Оказалось, что множество решений линейных уравнений в спинорном пространстве содержит функции, которые связаны с функциями Грина двух частиц в пространстве Минковского.

Я. ПАХНЕР, **НЕКОТОРЫЕ ВОПРОСЫ РЕЛЯТИВИСТСКОЙ КОСМОЛОГИИ** стр. 789—794

В работе дискутируются предположения, на которых в своей предыдущей работе автор обосновывает свою теорию релятивистской космологии [1]. Выведена простая формула, при использовании которой можно, опираясь на эмпирические данные, решить следующий вопрос: находится-ли вселенная в состоянии колебаний или-же она монотонически расширяется?

В дальнейших частях работы рассматриваются разные формы существования материи во вселенной.

В. ЖУРАВСКИЙ. **ПОЛОЖЕНИЯ И ИНТЕНСИВНОСТИ АБСОРБЦИОННЫХ ПОЛОС ДЛЯ КАТИОНОВ ТРИПАФЛАВИНА И ФЛУОРОСЦЕИНА** стр. 795—797

При применении метода FEMO были вычислены положения и интенсивности осцилляторов абсорбционных полос для катионов трипафлавина и флуоросцеина. Что касается положений, то полученные результаты находятся в хорошем согласии с экспериментальными данными.

Кроме полос, наблюдаемых уже ранее, получен ряд других полос, для которых соответствующие экспериментальные данные до настоящего времени отсутствуют.

Е. ЮЗЬКОВ, **О МОДИФИКАЦИИ МЕТОДА FEMO В СЛУЧАЕ БЕНЗОФЕНОНА** . . . . . стр. 799—801

Произведена модификация метода FEMO в применении к молекуле бензофенона путем введения потенциала, учитывающего неравносильность связей в скелете. Полученные результаты для частоты переходов, а в особенности для сил осциллятора значительно лучше согласуются с опытом, чем в случае применения обыкновенного метода FEMO. Это свидетельствует о том, что полученные волновые функции правильно описывают движение электронов π вдоль скелета молекулы.

Г. ЛЮКЕРСКИЙ, **ПОЛНАЯ СИММЕТРИЯ В СЛАБЫХ ВЗАИМОДЕЙСТВИЯХ** . . . . . стр. 803—806

В работе вводится универсальный нейтриновый заряд N. Предполагается, что он описывает источники слабых взаимодействий. Этот заряд геометризован при помощи трансформации типа Нишиджимы. Выражаем его при помощи электрического заряда Q и лептового числа L в виде формулы: $N = Q + L$.

Лагранжиан слабых воздействий при учете только шести полей, выступающих в модели Маркова — Сакаты, является инвариантным относительно симметрии Гамбы — Окубо — Маршака и дополнительного замещения $e \leftrightarrow \mu$.

М. СВЕНЦКИЙ и М. ШАЛЕК, **РАССЕЯНИЕ МЕЗОНОВ K^+ НА ПРОТОНАХ В СЛУЧАЕ СКАЛЯРНОГО СОПРЯЖЕНИЯ** . . . . . стр. 807—810

В работе обсуждается проблема рассеяния мезонов K^+ на протонах в приближении Тамма-Данкоффа. Предполагается, что сопряжение между K-мезонами и барионами скалярное. Представлены: сдвиг фазы $S_{1\,2}$, а также и общее сечение. Волны $P_{1\,2}$ и $P_{3/2}$ рассматриваются в приближении низких энергий. Рассматривается возможность появления резонанса $S_{1,2}$.

Полученные результаты не согласуются с экспериментальными данными, что приводит к заключению, что сопряжение не скалярное.

У. ШМИТ и А. ВЖЕСИНСКАЯ, **ИССЛЕДОВАНИЯ НАД ВЛИЯНИЕМ СТЕПЕНИ КОНЦЕНТРАЦИИ АКТИВАТОРА НА ГЛУБИНУ ЭЛЕКТРОННЫХ ЛОВУШЕК В ZnS** . . . . . стр. 811—816

В исследованиях над влиянием степени концентрации активатора на глубину электронных ловушек был применен метод кривых термолюминесценции на серии фосфоров ZnS-Cu. Были получены, хотя и небольшие, но постоянные перемещения максимумов этих кривых в сторону низких температур по мере возрастания степени концен-

трации активатора. Эти перемещения были констатированы так по отношению к максимумам присущим ловушкам, связанным с активатором (Cu), как и по отношению к максимумам присущим ловушкам, не имеющим связи с медью.

В работе обсуждаются следующие проблемы:

1) по какой причине изменения в степени концентрации меди вызывают изменения в глубине всех ловушек;

2) по какой причине эти изменения невелики по сравнению с изменениями (известными из литературных данных) глубины примесных уровней, наблюдаемых в исследованиях электрических свойств полупроводников.

А. КАВСКИЙ и З. ПОЛЯЦКИЙ, **КОНЦЕНТРАЦИОННОЕ ТУШЕНИЕ ФЛУОРЕСЦЕНЦИИ АНТРАЦЕНА В ПЛЕКСИГЛАССЕ** . . . . . стр. 817—819

Исследована концентрационная зависимость относительного выхода фотолюминесценции антрацена в плексиглассе.

Констатировано малое тушение флуоресценции антрацена в плексиглассе (большая половинная концентрация). В связи с этим ранее полученные результаты измерений анизотропии эмиссии в зависимости от концентрации необходимо сопоставить с теорией концентрационной деполяризации А. Яблонского для среднего радиуса активной сферы $R = 45$ Å, полученного для таких концентраций антрацена, при которых еще не выступает концентрационное тушение.

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